Problems for Stat. Mech. (Jan. 2007)

1. (30 min. Basic statistics)
You are playing the game "Rock, Scissors, Paper (RSP)" with your friend. At each run of the game, repeat RSP until either you or your friend wins. The winner of each run goes one step up in a staircase but the loser stays at the same level. (Looking back your childhood, you probably remember this game.) After N runs, compute the expectation value $<d>$ of the distance (the number of stpes) d between you and your friend. What is $<d^2>-<d>^2$?

2. (20 min. Basic statistics)
Consider an imaginary wall dividing this classroom into two rooms, A and B, of equal sizes. If there is only one oxygen molecule, the probability that you will not have the molecule is 1/2. Assume that the total number of the molecule is N.
(a) Compute the probability that the room A is completely empty and does not have any oxygen molecule. Explain why you are still alive.
(b) N_A and N_B are number of molecules in the room A and B, respectively. Compute $\frac{\sqrt{<N_A^2>-<N_A>^2}}{<N_A>}$ and show that it goes to zero in thermodynamic limit.

3. (30 min. . Equipartition theorem)
Assume that the atmosphere of the earth is solely composed of N_2 diatomic molecules. Estimate the temperature scale at which the root-mean-square velocity of the molecule becomes larger than the escape velocity for the gravitational field of the earth. Use g = 10m/s^2, proton mass = 1.7 x 10^{-27} kg, radius of the earth = 6400km, Boltzmann constant = 1.4 x 10^{-23} J/K.

4. (20 min. Thermodynamics)
A. The Gibbs free energy G is defined as $G=E-TS+pV$,
where the internal energy $E=E(S,V)$.
Show that G is a function of T and p, and derive a Maxwell relation
$\left(\frac{\partial S}{\partial p}\right)_T=-\left(\frac{\partial V}{\partial T}\right)_p$.
B. The Helmholtz free energy F is defined as $F=U-TS$
where the internal energy $U=U(S,V)$.
Show that F is a function of T and V, and derive a Maxwell relation
$\left(\frac{\partial S}{\partial V}\right)_T=\left(\frac{\partial p}{\partial T}\right)_V$.

5. (30 min. Thermodynamics)

Assume that a system composed of two different phases (1 and 2) is in equilibrium.
(a) Use the energy minimum principle to show that two phases have the same temperature, pressure, and chemical potential, i.e., $T^{(1)} = T^{(2)}, p^{(1)} = p^{(2)}, \mu^{(1)} = \mu^{(2)}$
What happens if you have $\mu^{(1)} > \mu^{(2)}$?
(b) Require the stability and show that the specific heat $c_V \geqq 0$.

6. (30 min. Canonical partition function of specific systems)
A. Consider a spin-half system of N noninteracting spins in a magnetic field. The Hamiltonian is given by $H = -J\sum_i S_i$, where the i-th spin S_i can have value ± 1 and J is a some constant with the dimension of energy.
(a) Use the microcanonical ensemble to get $U(T)$.
(b) Use the canonical ensemble to get $U(T)$ and the magnetic susceptibility $\chi(T)$.

B. Consider a system of N noninteracting spins in a magnetic field. The Hamiltonian is given by $H = -J\sum_i S_i$, where the i-th spin S_i can have value $-1,0,1$ and J is a some constant with the dimension of energy. Use the canonical ensemble to get the magnetization $m(T) = (1/N) < \sum S_i >$.

C. Consider a two-level system (energy levels are $-\epsilon$ and ϵ) composed of N noninteracting distinguishable particles. Calculate the Helmholtz free energy, the entropy, and the specific heat as functions of temperature T.

7. (30 min. Canonical partition function of specific systems)
Consider the system of N one-dimensional harmonic oscillators.
(a) The energy eigenvalues are written as $E_n = \hbar\omega(n + 1/2)$. Obtain the canonical partition function Z.
(b) Compute the internal energy U from the partition function and show that $U = Nk_BT$ in the classical limit.

8. (20 min. Variational method for ensembles.)
A. Consider the grandcanonical ensemble. Use entropy maximum principle applied to the expression of the entropy
$$S = -k_B \sum_\nu P_\nu \log P_\nu$$
with the constraints

$< E > = \sum_{\nu} E_{\nu} P_{\nu},$
$< N > = \sum_{\nu} N_{\nu} P_{\nu}$
$\sum_{\nu} P_{\nu} = 1,$

and obtain the grandcanonical partition function. Explain how you can get $< E >$ and $< N >$ from the partition function.

B. Consider canonical ensemble. Use the entropy maximum principle applied to the expression of the entropy

$S = -k_B \sum_{\nu} P_{\nu} \log P_{\nu}$

with the constraints

$< E > = \sum_{\nu} E_{\nu} P_{\nu},$
$\sum_{\nu} P_{\nu} = 1,$

and obtain the canonical partition function. Explain how you can get $< E >$ from the partition function.

9. (30 min. Ensembles and fluctuation.)

A. Consider the grandcanonical ensemble.

(a) Show the probability of finding the system in a particular microstate j is given by

$$P_j = \frac{\exp(-\beta E_j + \beta\mu N_j)}{Z_G}$$

with the grand partition function

$$Z_G = \sum_j \exp(-\beta E_j + \beta\mu N_j)$$

(b) Find the connection between Z_G and a thermodynamic potential.

(c) Show that

$< (\Delta N)^2 > = N\frac{k_B T \kappa_T}{v}$ with the specific volume v, and explain the concept of the ensemble equivalence.

B. Consider the canonical ensemble

(a) Show the probability of finding the system in a particular microstate j is given by

$$P_j = \frac{\exp(-\beta E_j)}{Z}$$

with the grand partition function

$$Z = \sum_j \exp(-\beta E_j)$$

(b) Find the connection between Z and the Helmholtz free energy

(c) Show that

$< (\Delta E)^2 > = < E^2 > - < E >^2 = Nk_B T^2 c_V$ with the specific heat $c_V$at constatnt volume and explain the equivalence between canonical and microcanonical ensembles.

10. (30 min. Partition function for quantum particles)
Consider a two-particle system. Each particle can have energy 0 and Δ. Compute the partition function for the following cases:
(a) When particles are distinguishable.
(b) When particles are fermions.
(c) When particles are bosons.

11. (30 min. Ideal quantum gases)
Use the grandcanonical ensemble and calculate the expectation value of the occupation number for
(a) Bose-Einstein statistics
(b) Fermi-Dirac statistics,
and
(c) show that in the classical limit, both expressions approach the same.

12. (20 min. Ideal quantum gas)
Use the grandcanonical ensemble for ideal fermions to show that $U = \frac{3}{2} pV$. Compute the same for classical ideal gas.

13. (20 min. Fermi energy.)
A. Compute the Fermi energy E_F for two-dimensional ideal spin-1/2 fermions with the particle areal density n.
B. Show that the Fermi energy E_F of the three-dimensional ideal gas of spin 1/2 fermions is written as $E_F = (\hbar^2/2m)(3\pi^2 n)^{2/3}$, where n is the particle density.

14. (20 min. Photon gas & Planck law)
For a photon gas with $\omega(k) = ck$, compute the canonical partition function and obtain the expression for the internal energy. Compute the spectral density of energy $u(\nu)$ with the frequency ν. Take the classical limit to get the Rayleigh-Jeans law.

15. (20 min. Phonon & Debye approximation)
Apply the Debye approximation to D-dimensional phonon problem, and show that the specific heat in the low-temperature limit is proportional to T^D.

16. (30 min. Bose-Einstein condensation)

(a) (150 pts) Show that the ground state occupancy of ideal bosons N_0 is calculated as

$$N_0 = N\left[1-\left(\frac{T}{T_C}\right)^{3/2}\right],$$

with the Bose-Einstein critical temperature T_c.

(b) (50 pts) Find the expression of T_c in terms of number density per volume n, the Plank constant h , mass of the boson m, and the Boltzmann constant k_B.

17. (20 min. Concepts in Stat. Mech.)

Explain very briefly the following concepts (you may answer in Korean):

(a) equilibrium state
(b) entropy
(c) microcanonical ensemble
(d) canonical ensemble
(e) ergodic hypothesis
(f) fundamental postulate of statistical mechanics (postulate of equal *a priori* probabilities)
(g) extensive quantities and intensive quantities
(h) phase transition

18. (20 min. Concepts in Stat. Mech.)

Explain the following concepts very briefly (Use of equations and long descriptions are discouraged):

(a) Ralyeigh-Jeans law and ultraviolet catastrophe
(b) Pauli paramagnetism
(c) Landau diamagnetism
(d) Spin wave and magnon

19. (20 min. Concepts in Stat. Mech.)

A. Explain very briefly why the time irreversibility in your everyday experience does not contradict to the time-reversal symmetry in Newtonian mechanics.

B. You can see glass windows break into small fragments but you do not see those fragments reassemble spontaneously to become a window. Explain why.

20. (30 min. Transfer matrix method)

Consider the one-dimensional spin model with $S_i = \pm 1$ in the presence of external magnetic field B:

$$H = -J\sum_i S_i S_{i+1} - B\sum_i S_i \ .$$

Use the periodic boundary condition, $S_{i+N} = S_i$, and obtain the partition function and the magnetization as a function of temperature.

21. (30 min. Mean-field approximation. Difficult for master students)
Use the mean-field approximation for the Ising model in D-dimension, and find the critical exponents α and γ for the specific heat $C \propto |T - T_c|^{-\alpha}$ and the magnetic susceptibility $\chi \propto |T - T_c|^{-\gamma}$, respectively.

22. (30 min. Mean-field approximation)
Apply the mean-field approximation to the Ising model on the bcc (body-centered cubic) structure. What is the critical temperature?

23. (30 min. Mean-field approximaion)
Obtain the entropy S of the mean-field Ising model as a function of
(a) the total magnetization M
(b) the total energy E
(c) the temperature T

24. (20 min.. Phase transition in 1D Ising)
Show that the 1D Ising model cannot have a ferromagnetic phase transition at nonzero temperature.

25. (20 min.. RG)
Explain very briefly why the 2D Ising model has three fixed points in Renormalization group (RG) flow, and discuss the meaning of each RG fixed point.