EM PROBLEM SET 5
2019 — SOLUTIONS

Q1. $\vec{\nabla}\cdot\vec{B} = 0$ always $\Rightarrow$ no inward or outward flux of the magnetic field $\Rightarrow$ no radial component to the $\vec{B}$ field, it circulates around the current.

Symmetry $\Rightarrow$ magnitude of $\vec{B}$ depends only on the distance r from the centre of the wire.

For $r<R$: choose a circular loop of radius r as shown:

Since the wire has radius R, and the current I is uniformly distributed through the wire, the current density j in the wire (current per unit cross-sectional area)

So $\quad j = \dfrac{I}{\pi R^2}$

So the current through the loop of radius r

$= $ (current density) $\times$ (area of loop)

$$= \frac{I}{\pi R^2} \times \pi r^2$$

$$= I \left(\frac{r}{R}\right)^2$$

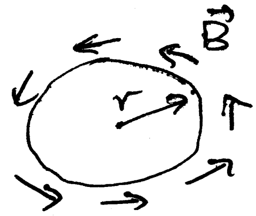
Ampere's law:

(circulation of $\vec{B}$ around loop)

$= \dfrac{1}{\epsilon_0 c^2}$ (current through loop)

Circulation of $\vec{B}$ around loop

$$= \oint \vec{B}(\vec{r}).d\vec{l}$$



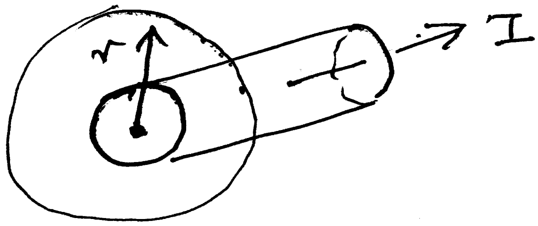
$$= B(r) \,.\, 2\pi r$$

So Ampere's law

$$\Rightarrow \quad B(r) \,.\, 2\pi r = \frac{1}{\epsilon_0 c^2} I \left(\frac{r}{R}\right)^2$$

$$\Rightarrow \quad B(r) = \frac{I r}{2\pi \epsilon_0 c^2 R^2}$$

<u>For $r > R$</u>: Choose the Amperian loop to be a circle of radius r outside the wire:



Current enclosed by loop $= I$

Circulation of $\vec{B}$ around loop

$$= B(r) \cdot 2\pi r$$

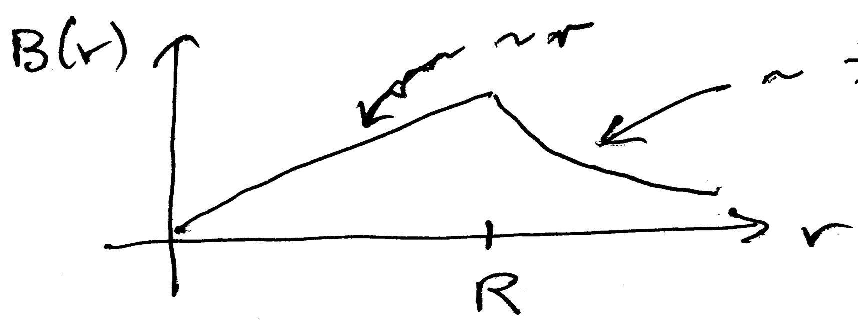
Ampere's law:

(circulation of $\vec{B}$ around loop)

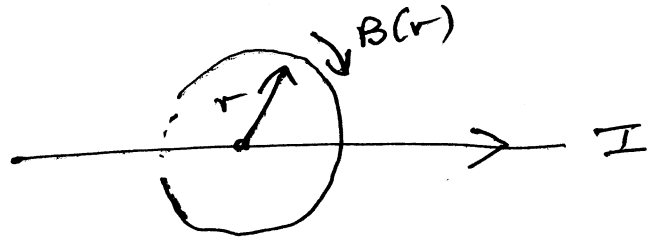
$$= \frac{1}{\varepsilon_0 c^2} \text{ (current through loop)}$$

$$\Rightarrow \quad B(r) \cdot 2\pi r = \frac{I}{\varepsilon_0 c^2}$$

$$\Rightarrow \quad B(r) = \frac{I}{2\pi \varepsilon_0 c^2 r}$$



2 (a)



Ampere's law: (circulation of $\vec{B}$ around loop) $= \frac{1}{\varepsilon_0 c^2}$ (current through loop)

$$\Rightarrow \quad B(r)\, 2\pi r = \frac{I}{\varepsilon_0 c^2}$$

$$\Rightarrow \quad B(r) = \frac{I}{2\pi \varepsilon_0 c^2 r}$$

(b). The infinitesimal strip between r and $r+dr$ has an area $a\,dr$

$\Rightarrow$ the magnetic flux through the strip is

$$d\Phi = B(r) \times (\text{area of strip})$$

$$= B(r)\, a\, dr$$

$$= \frac{I}{2\pi \varepsilon_0 c^2 r} \cdot a \cdot dr$$

(c) To get the total magnetic flux through the loop, we have to "sum" the magnetic fluxes through all the infinitesimal strips from $r = d$ to $r = d+a$

$$\Phi = \int_{r=d}^{r=d+a} d\Phi$$

$$= \int_{d}^{d+a} \frac{I}{2\pi \varepsilon_0 c^2 r} \cdot a \cdot dr$$

$$= \frac{I a}{2\pi \varepsilon_0 c^2} \int_{d}^{d+a} \frac{dr}{r}$$

$$= \frac{I a}{2\pi \varepsilon_0 c^2} \left[\ln r \right]_{d}^{d+a}$$

$$= \frac{I a}{2\pi \varepsilon_0 c^2} \left(\ln (d+a) - \ln a \right)$$

$$= \frac{I a}{2\pi \varepsilon_0 c^2} \ln \left(\frac{d+a}{d} \right)$$

$$= \frac{I a}{2\pi \varepsilon_0 c^2} \ln \left(1 + \frac{a}{d} \right)$$

(d) Using the right hand rule, the magnetic field inside the loop due to the wire is <u>out of the page</u>.

The flux through the loop is

$$\Phi = \frac{I a}{2\pi \varepsilon_0 c^2} \ln\left(1 + \frac{a}{d}\right),$$

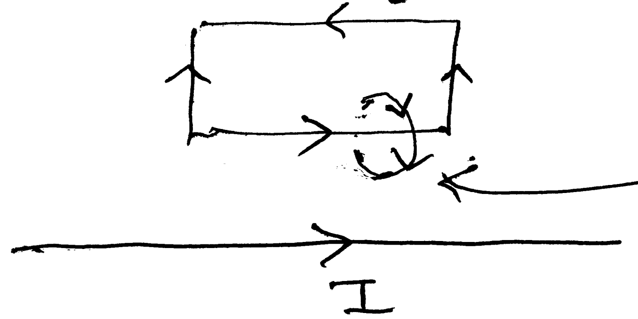
which <u>decrease</u> as d increase (as $\frac{a}{d}$ decrease).

Lenz's law says that the induced current will produce a magnetic field which will <u>oppose</u> the <u>decrease</u> in the flux of the magnetic field <u>out</u> of page

ie. the induced current must produce a flux <u>out of</u> the page.

Using the right hand rule:

direction of induced current



magnetic field produced by induced current

(produces flux out of page, opposing the <u>decrease</u> in flux out of page due to expansion of size of current loop)

(e) Flux through loop as function of time

$$\Phi(t) = \frac{I a}{2\pi \varepsilon_0 c^2} \ln\left(\frac{d(t)+a}{d(t)}\right)$$

$$= \frac{I a}{2\pi \varepsilon_0 c^2}\Big(\ln(d(t)+a) - \ln d(t)\Big)$$

$$\mathcal{E} = \frac{d}{dt}\Phi(t)$$

$$= \frac{I a}{2\pi \varepsilon_0 c^2}\left(\frac{1}{(d(t)+a)}\frac{d}{dt}(d(t)) - \frac{1}{d(t)}\frac{d}{dt}(d(t))\right)$$

$$= \frac{I a}{2\pi \varepsilon_0 c^2} \left(\frac{v}{d(t)+a} - \frac{v}{d(t)} \right)$$

where $v = \frac{d}{dt}(d(t))$ is the speed at which the loop is moving.