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## 0.1 Completing Ampere's law - the "displacement current"

Up until now, we have been using a version of Ampere's law applicable to magnetostatics (time-independent charge and current densities which means time-independent electric and magnetic fields):

$$\vec{\nabla} \times \vec{B}(\vec{r}) = \frac{1}{\epsilon_0 c^2}\,\vec{j}(\vec{r}).$$

However, once we introduce time dependence, an additional term called the "displacement current" must be included in Ampere's law.

To understand why we need to modify Ampere's law, let's go back an reconsider an analysis we did earlier. We considered a closed surface $S$ enclosing a volume $V$. The net flow of current through the closed surface $S$ is

$$\begin{aligned}\oint_S \vec{j}(\vec{r}) \cdot d\vec{S} &= \epsilon_0 c^2 \oint_S (\vec{\nabla} \times \vec{B}(\vec{r})) \cdot d\vec{S} \quad \text{(using Ampere's law)} \\ &= \int_V d^3\vec{r}\; \vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}(\vec{r})) \quad \text{(using Gauss's law)} \\ &= 0\end{aligned}$$

as a result of the vector identity $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{V}(\vec{r}))$ for any vector field $\vec{V}(\vec{r})$. The fact that the net flow of current through any closed surface $S$ is zero is consistent with the fact that in magnetostatics, we do not consider "sources" or "sinks" for currents (such as battery terminals), so the net flow of current into a closed surface must match the net flow outward, and the total charge inside the surface does not change with time.

However, in electrodynamics, we do allow for the existence of current sources and sinks, so that currents can flow outward or inward through the closed surface, and the charge enclosed by the surface changes with time (since currents are flows of charges). The statement $\oint_S \vec{j}(\vec{r}) \cdot d\vec{S} = 0$ applicable to magnetostatics must be replaced by the more general statement

$$\oint_S \vec{j}(\vec{r}, t) \cdot d\vec{S} = -\frac{d}{dt}\int_V d^3\vec{r}\,\rho(\vec{r}, t),$$

where $V$ is the volume enclosed by the surface $S$ and $\rho(\vec{r}, t)$ is the charge density. This is simply a statement of conservation of charge. The integral $\int_V d^3\vec{r}\,\rho(\vec{r}, t)$ is the total charge inside the surface $S$. If there is a net flow of current out through the surface $S$, the left hand side of this equation is nonzero and positive; it means that charge is flowing out of the volume $V$, and so the total charge $\int_V d^3\vec{r}\,\rho(\vec{r}, t)$ inside $V$ must decrease with time i.e. $\frac{d}{dt}\int_V d^3\vec{r}\,\rho(\vec{r}, t)$ is negative. The minus sign in the equation is there because when current flows out through $S$, the charge inside $S$ decreases; and when current flows in through $S$, the charge inside $S$ increases.

Using Gauss's law,

$$\oint_S \vec{j}(\vec{r}, t) \cdot d\vec{S} = \int_V \vec{\nabla} \cdot \vec{j}(\vec{r}, t)\, d^3\vec{r},$$

and so the equation stating conservation of charge becomes

$$\int_V \vec{\nabla} \cdot \vec{j}(\vec{r}, t)\, d^3\vec{r} = -\frac{d}{dt} \int_V \rho(\vec{r}, t)\, d^3\vec{r}.$$

Removing the integration on both sides, we get

$$\vec{\nabla} \cdot \vec{j}(\vec{r}, t) = -\frac{\partial \rho(\vec{r}, t)}{\partial t}.$$

This is called the *continuity equation*, and is an equivalent statement of charge contribution: convergence or divergence of the current at a point means the charge at that point must be changing.

So can now recognize that in electrodynamics, our version of Ampere's law that was perfectly good for magnetostatics is not consistent with conservation of charge. It was Maxwell who realized that an additional term is required in Ampere's law in the time-dependent case, and the full law reads:

$$\vec{\nabla} \times \vec{B}(\vec{r}, t) = \frac{1}{\epsilon_0 c^2} \vec{j}(\vec{r}, t) + \frac{1}{c^2} \frac{\partial \vec{E}(\vec{r}, t)}{\partial t}.$$

As required, the extra term disappears when we have no time-dependence. For historical reasons, this extra term is called *"Maxwell's displacement current."* However, this is a bad choice of terminology - it has nothing to do with electric currents. **It simply encompasses the fact that in addition to being produced by electric currents, magnetic fields can also be created by time-varying electric fields.** The factor of $\frac{1}{c^2}$ means the magnetic field produced by a time-varying electric field is usually very small - nevertheless, this term is essential to ensure charge conservation, and we will see it is very important in the understanding of electromagnetic radiation.

To see how the addition of the "Maxwell displacement current" yields conservation of charge in a case where we have sources and sinks of currents (and so net current flow through closed surfaces), take the divergence of Ampere's law and use the vector identity $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}(\vec{r}, t)) = 0$, and we get

$$0 = \frac{1}{\epsilon_0 c^2} \vec{\nabla} \cdot \vec{j}(\vec{r}, t) + \frac{1}{c^2} \frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{E}(\vec{r}, t)).$$

Using Gauss's law $\vec{\nabla} \cdot \vec{E}(\vec{r}, t) = \frac{\rho(\vec{r}, t)}{\epsilon_0}$, this becomes

$$0 = \frac{1}{\epsilon_0 c^2} \vec{\nabla} \cdot \vec{j}(\vec{r}, t) + \frac{1}{\epsilon_0 c^2} \frac{\partial \rho(\vec{r}, t)}{\partial t}.$$

Removing the factor $\frac{1}{\epsilon_0 c^2}$, we obtain the continuity equation, which was the statement of charge conservation.

## 0.2 Example showing the need for the displacement current

Consider a surface $S$ with boundary the closed loop $\Gamma$. Then the circulation of the magnetic field around the loop is :

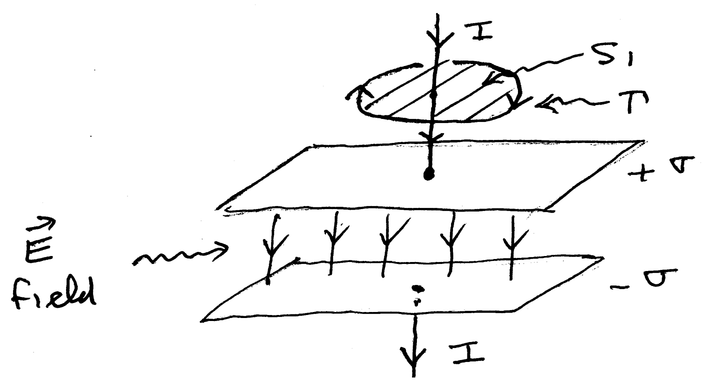
$$\oint_\Gamma \vec{B}\cdot d\vec{\ell} = \int_S \vec{B}\cdot d\vec{S} \quad \text{(using Stokes' theorem)}.$$

If we apply Ampere's law in the form $\vec{\nabla}\times\vec{B} = \frac{1}{\epsilon_0 c^2}\vec{j}$,

$$\oint_\Gamma \vec{B}\cdot d\vec{\ell} = \frac{1}{\epsilon_0 c^2}\int \vec{j}\cdot d\vec{S},$$

where the right hand side is the current through the surface $S$. This result for the circulation of the magnetic field around $\Gamma$ applies for any surface $S$ with boundary $\Gamma$.

Let's apply this result to the situation of charging up a capacitor - adding charge to the top plate via a current $I$and removing charge from the bottom plate by a current of the same magnitude, and consider the surface $S_1$ shown with boundary $\Gamma$:




Then the circulation of the magnetic field around $\Gamma$ is

$$\oint_\Gamma \vec{B}\cdot d\vec{\ell} = \frac{1}{\epsilon_0 c^2}\,(\text{current through } S_1) = \frac{I}{\epsilon_0 c^2}.$$

So as expected, we find a nonzero circulation of the magnetic field around the current.

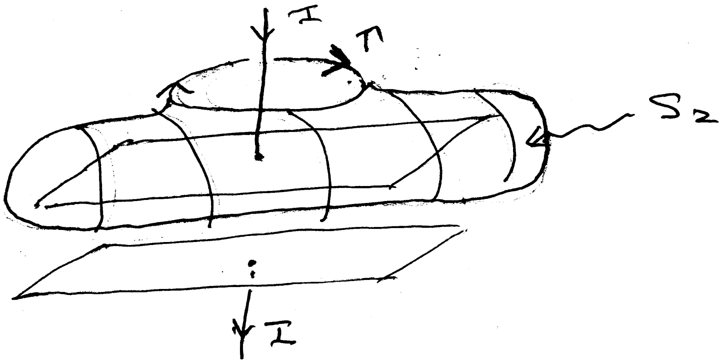
However, for the same loop $\Gamma$, suppose we choose the surface $S_2$ shown below:

In this case, there is no current through $S_2$, and so Ampere's law in the form used will yield the value zero for the circulation of the magnetic field around the current, which is clearly incorrect. What has gone wrong? The problem is that for the choice of surface $S_2$, the displacement current makes a nonzero contribution: using the full form of Ampere's law

$$\vec{\nabla}\times\vec{B} = \frac{1}{\epsilon_0 c^2}\vec{j} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t},$$

the circulation of the magnetic field around $\Gamma$ is

$$\oint_\Gamma \vec{B}\cdot d\vec{\ell} = \frac{1}{\epsilon_0 c^2}\,(\text{current through } S_2) + \frac{1}{c^2}\frac{d}{dt}\int_{S_2} \vec{E}\cdot d\vec{S}.$$




Whilst the first term on the right hand side still vanishes, the second term is non-vanishing as an electric field builds up between the plates of the capacitor as the capacitor charges, consistent with the fact that the circulation of the magnetic field around the current $I$ is nonzero. Indeed, the electric field between the capacitor plates is $\frac{\sigma}{\epsilon_0}$, where $\sigma$ is the charge density on the plates. I the area of the capacitor plates is $A$, then $\int_{S_2} \vec{E} \cdot d\vec{S} = \frac{\sigma A}{\epsilon_0} = \frac{Q}{\epsilon_0}$, where $Q$ is the charge on the plate. So

$$\frac{1}{c^2} \frac{d}{dt} \int_{S_2} \vec{E} \cdot d\vec{S} = \frac{1}{\epsilon_0 c^2} \frac{dQ}{dt} = \frac{I}{\epsilon_0 c^2},$$

which is the correct value of the circulation of the magnetic field around the current.

# 1 Electromagnetic Waves

Consider Gauss's law

$$\vec{\nabla} \cdot \vec{E}(\vec{r}, t) = \frac{\rho(\vec{r}, t)}{\epsilon_0}.$$

In regions of space where the charge density vanishes,

$$\vec{\nabla} \cdot \vec{E}(\vec{r}, t) = 0.$$

This does not necessarily imply that the electric field vanishes in these regions. For example, for a point charge $q$ at the origin, the electric field takes the form

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\epsilon_0 r^2}\, \vec{e}_r,$$

where $\vec{e}_r$ is a unit vector in the radial direction. The charge density is nonzero only at the origin, but the electric field is everywhere nonvanishing.

Maxwell's equations in regions where $\rho(\vec{r}, t) = 0$ and $\vec{j}(\vec{r}, t) = 0$ are called the " vacuum Maxwell equations". There may or may not be electric and magnetic fields in these regions, depending on the arrangement of charges and currents outside the region. A very important class of solutions to the vacuum Maxwell equations correspond to electromagnetic waves (such as light, radio waves, microwaves, X-rays, gamma rays etc). The electric and magnetic fields in electromagnetic waves are usually created by oscillating charges in antennas.

The vacuum Maxwell equations are:

$$\vec{\nabla} \cdot \vec{E}(\vec{r}, t) = 0 \tag{1.1}$$

$$\vec{\nabla} \times \vec{E}(\vec{r}, t) = -\frac{\partial \vec{B}(\vec{r}, t)}{\partial t} \tag{1.2}$$

$$\vec{\nabla} \cdot \vec{B}(\vec{r}, t) = 0 \tag{1.3}$$

$$\vec{\nabla} \times \vec{B}(\vec{r}, t) = \frac{1}{c^2}\frac{\partial \vec{E}(\vec{r}, t)}{\partial t}. \tag{1.4}$$

Note that there is a symmetry: the equations are unchanged by the substitutions $\vec{E} \to -c\vec{B}$, $\vec{B} \to \frac{1}{c}\vec{E}$. This is known as duality symmetry.

The vacuum Maxwell equations are all first order partial differential equations. Consider first equations (1.2) and (1.4). They are coupled equations - they involve both $\vec{E}$ and $\vec{B}$. A common trick to decouple first order differential equations is to differentiate to make them second order differential equations. First we take the curl of both sides of (1.2), and use the vector identity

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{V}\right) = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{V}\right) - \vec{\nabla}^2 \vec{V},$$

where the Laplacian operator $\nabla^2 = \vec{\nabla} \cdot \vec{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. This gives

$$\vec{\nabla}\left(\vec{\nabla} \cdot \vec{E}(\vec{r}, t)\right) - \vec{\nabla}^2 \vec{E}(\vec{r}, t) = -\frac{\partial}{\partial t}\vec{\nabla} \times \vec{B}(\vec{r}, t).$$

The first term on the left hand side vanishes as a result of equation (1.1), and equation (1.4) means the right hand side is equal to $-\frac{1}{c^2}\frac{\partial^2 \vec{E}(\vec{r},t)}{\partial t^2}$. So we get the decoupled differential equation

$$\vec{\nabla}^2 \vec{E}(\vec{r},t) = \frac{1}{c^2}\frac{\partial^2 \vec{E}(\vec{r},t)}{\partial t^2}. \tag{1.5}$$

Similarly, taking the curl of both sides of equation (1.4) yields

$$\vec{\nabla}\left(\vec{\nabla}\cdot\vec{B}(\vec{r},t)\right) - \vec{\nabla}^2\vec{B}(\vec{r},t) = \frac{1}{c^2}\frac{\partial}{\partial t}\vec{\nabla}\times\vec{E}(\vec{r},t).$$

The first term on the left hand side vanishes using (1.3), and using (1.2) for the term on the right hand side, we arrive at

$$\vec{\nabla}^2 \vec{B}(\vec{r},t) = \frac{1}{c^2}\frac{\partial^2 \vec{B}(\vec{r},t)}{\partial t^2}. \tag{1.6}$$

This is a decoupled second order differential equation for the magnetic field, and takes the same form as the equation for the electric field (1.5).

The equations (1.5) and (1.6) are vector equations, and so we can consider the $x$, $y$ and $z$ components. They all take a similar form:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right) f(\vec{r},t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} f(\vec{r},t). \tag{1.7}$$

This is known as the *three-dimensional wave equation.*

Consider the one-dimensional wave equation:

$$\frac{\partial^2}{\partial x^2} f(x,t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} f(x,t). \tag{1.8}$$

This equation admits solutions of the form

$$f(x,t) = A\, e^{i(kx-\omega t)}$$

provided $\frac{\omega}{k} = c$, the speed of light. (Note: this is not the most general solution, it is a special type of solution related to sinusoidal waves).

**Check:**

$$\begin{aligned} \frac{\partial^2}{\partial x^2} f(x,t) &= (ik)^2 f(x,t) \\ \frac{1}{c^2}\frac{\partial^2}{\partial t^2} f(x,t) &= \frac{(-i\omega)^2}{c^2} f(x,t) \end{aligned}$$

These are equal if $k^2 = \frac{\omega^2}{c^2}$.

Using

$$A\, e^{i(kx-\omega t)} = A\cos(kx-\omega t) + i\,A\sin(kx-\omega t),$$

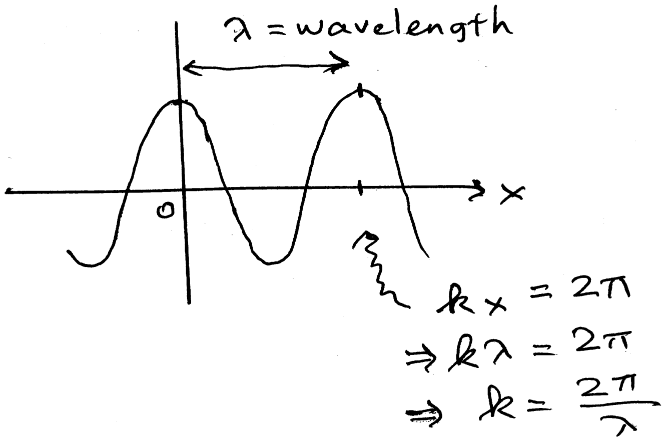
the real and imaginary parts of this solution represent sinusoidal waves, such as the displacement of a water surface. The wave motion is characterised by

$$\begin{aligned} k &= \text{wave number} = \frac{2\pi}{\lambda} \\ \omega &= \text{angular frequency} = 2\pi f, \end{aligned}$$

where $\lambda$ is the wavelength and $f$ is the frequency of the wave motion. The argument $(kx - \omega t)$ is called the *phase* of the wave at the point $x$ at time $t$. The ratio $\frac{\omega}{k}$ is known as the *phase velocity* of the wave, it determines the speed at which a point of given phase moves (and therefore the speed of the wave). In the case of the one-dimensional wave equation we are dealing with, $\frac{\omega}{k} = c$, the speed of light.

**Proofs of basic wave properties**

Consider the waveform $f(x,t) = A\cos(kx - \omega t)$. A snapshot of the waveform at $t = 0$ is $A\cos kx$ :




The displacement of the point at $x = 0$ as a function of time is $A\cos\omega t$: