Introduction to Robotics

ISS3180-01

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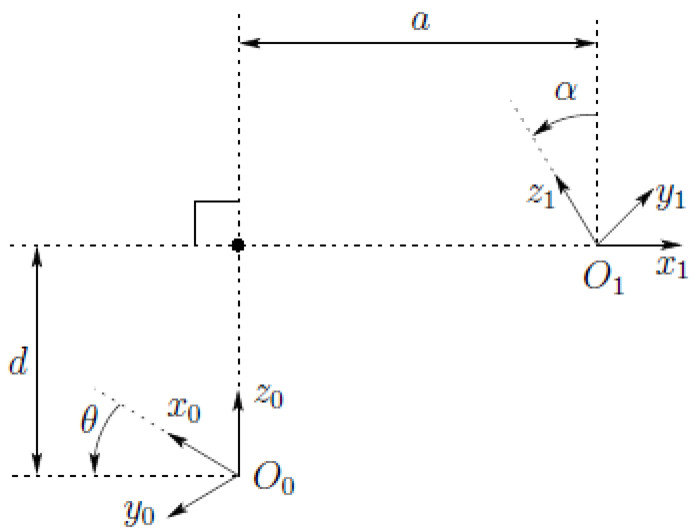
The physical basis for DH parameters



- a_i: $\overline{O_i O_{i-1}}$ @ x_i
- α_i: $\angle(z_{i-1} \rightarrow z_i)$ @ x_i
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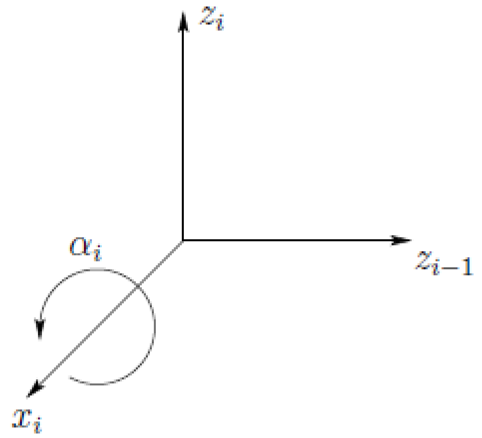

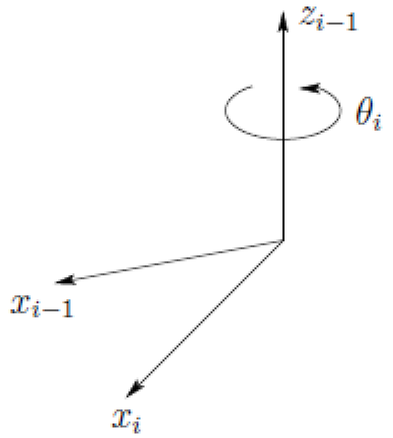

The Denavit-Hartenberg (DH) Convention

- Representing each individual homogeneous transformation as the product of four basic transformations:

$$
\begin{aligned}
{}^{i-1}_{i}H &= \mathrm{Rot}_{z,\theta_i}\mathrm{Trans}_{z,d_i}\mathrm{Trans}_{x,a_i}\mathrm{Rot}_{x,\alpha_i} \\
&= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} & 0 & 0 \\ s_{\theta_i} & c_{\theta_i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{\alpha_i} & -s_{\alpha_i} & 0 \\ 0 & s_{\alpha_i} & c_{\alpha_i} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

DH for Two-Link Manipulators

- PP robot
- DH frames
- DH table

- a_i: $\overline{O_i O_{i-1}}$ @ x_i
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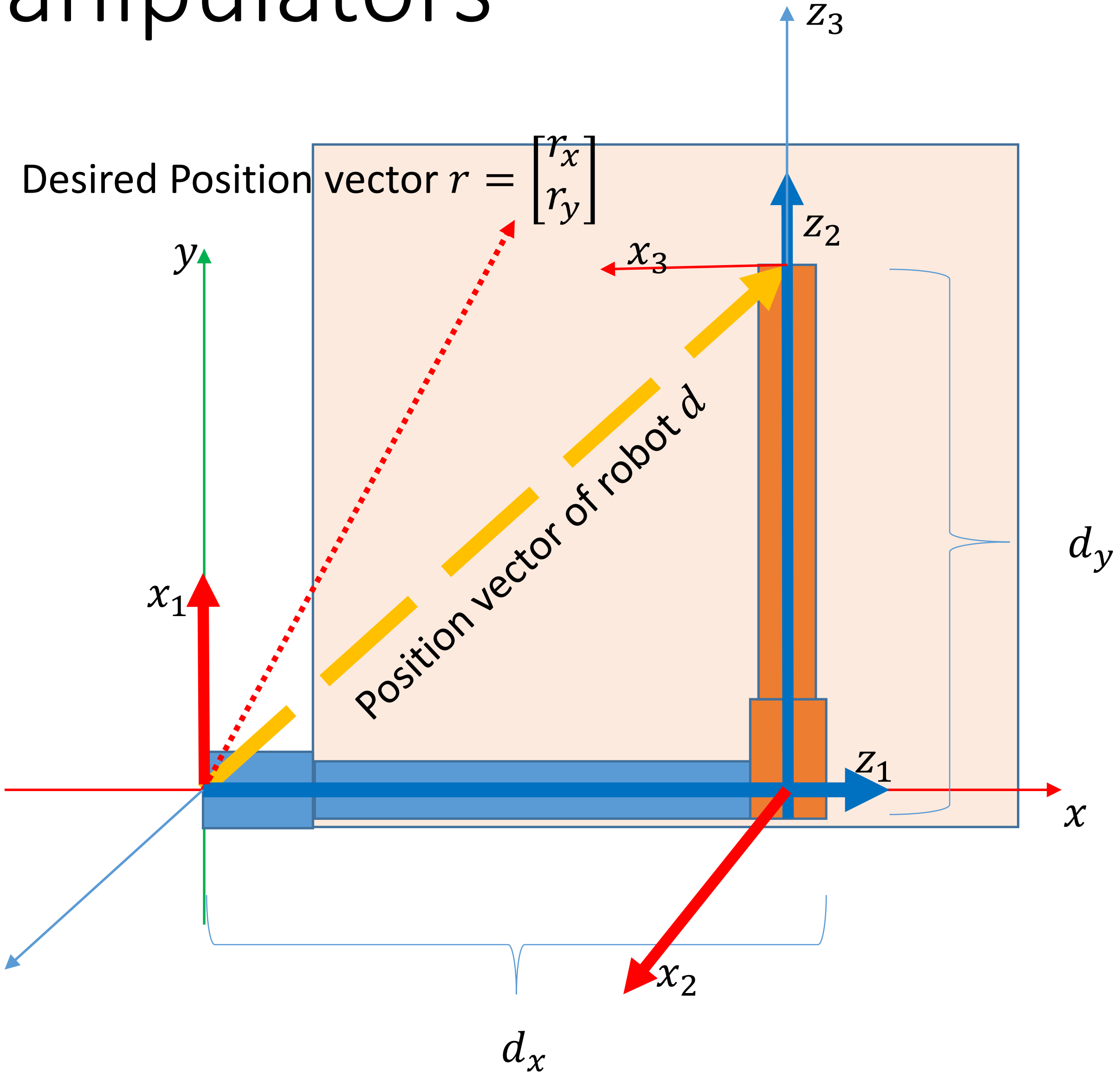

DH for Two-Link Manipulators

- PP robot
- DH frames
- DH table

i	α_i	a_i	θ_i	d_i
0	-	-	-	-
1				
2				
3				

- a_i: $\overline{O_i O_{i-1}}$ @ x_i
- α_i: $\angle(z_{i-1} \to z_i)$ @ x_i
- d_i: $\overline{O_i O_{i-1}}$ @ z_{i-1}
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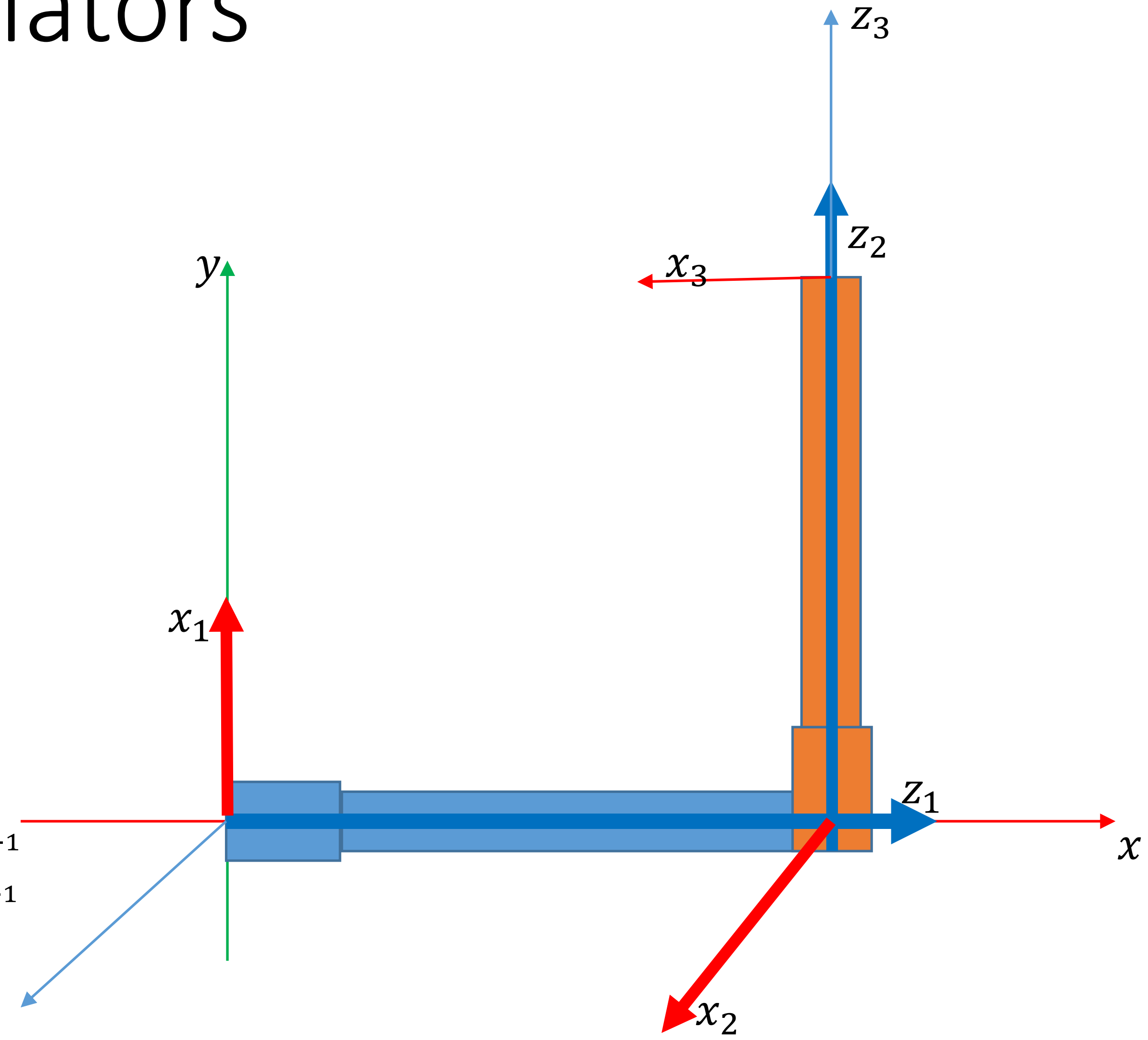

DH for Two-Link Manipulators

- RR robot
- DH frames
- DH table

- a_i: $\overline{O_i O_{i-1}}$ @ x_i
- α_i: $\angle(z_{i-1} \rightarrow z_i)$ @ x_i
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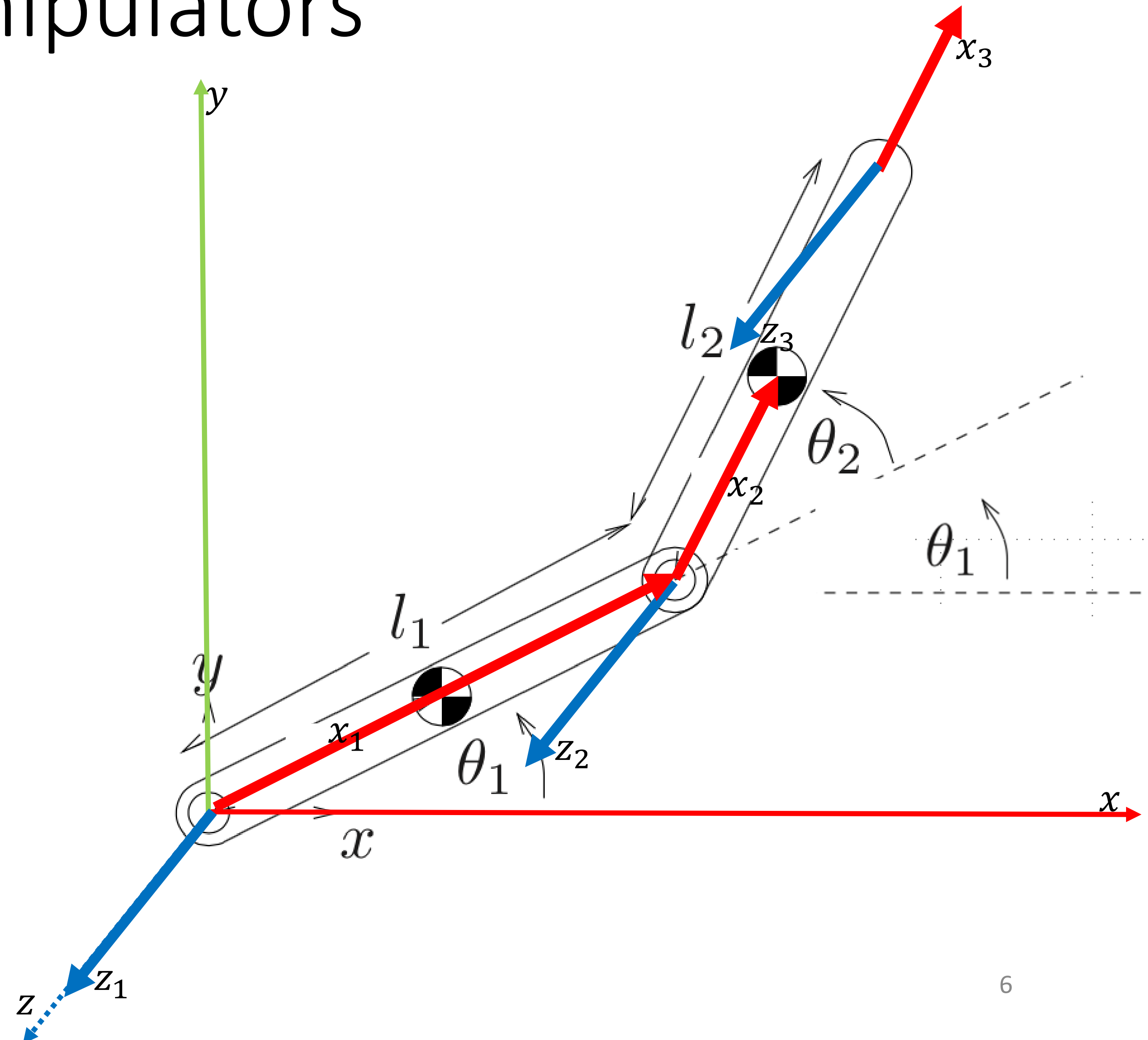

DH for Two-Link Manipulators

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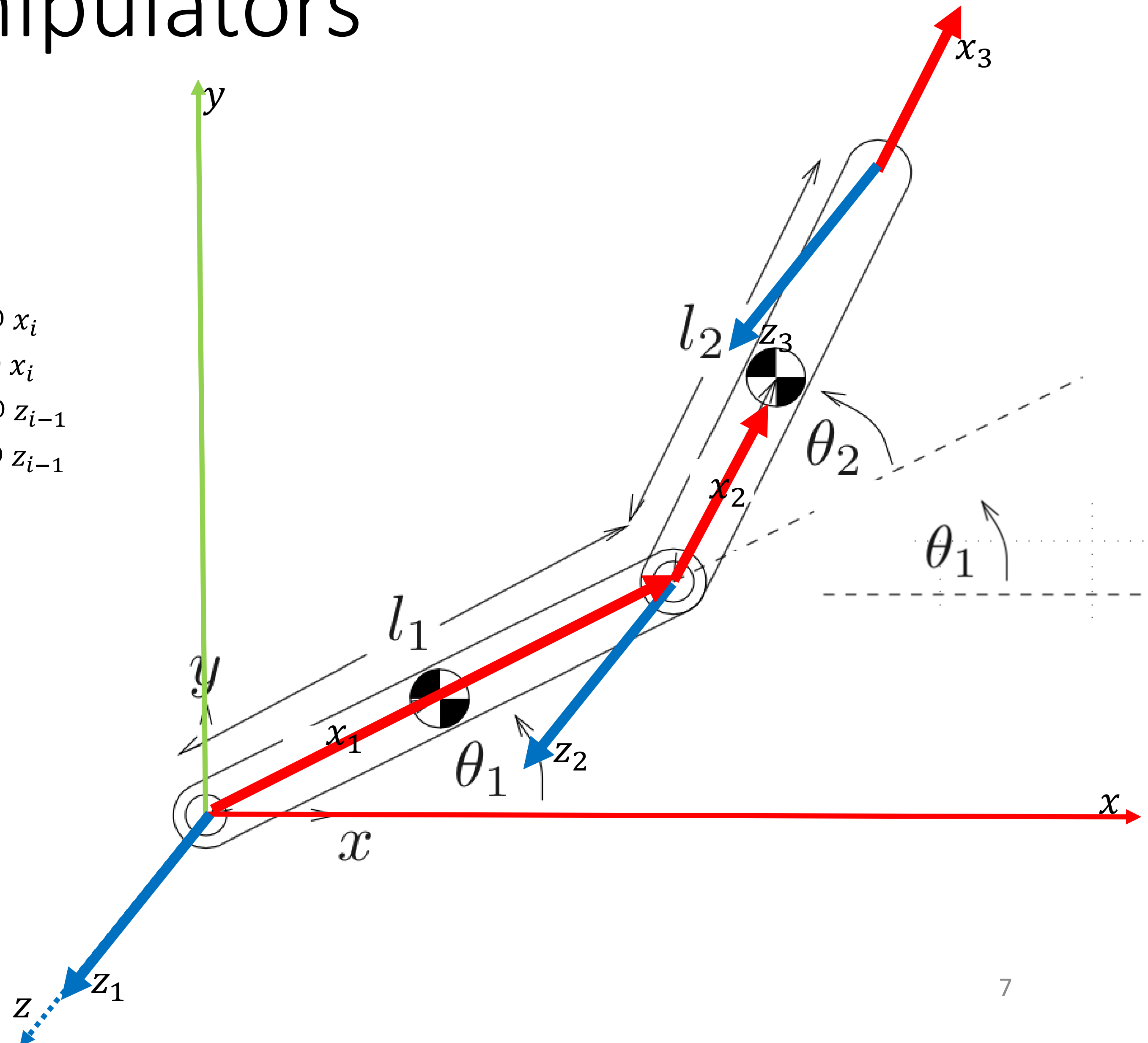
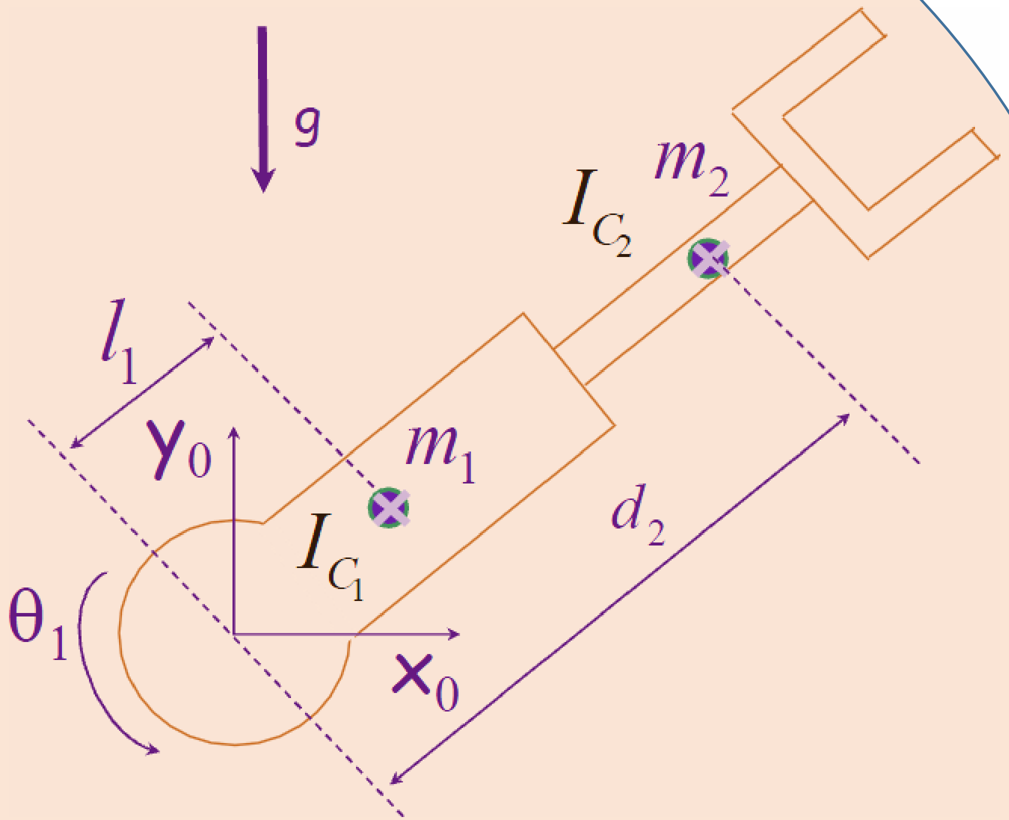
0	-	-	-	-
1				
2				
3				



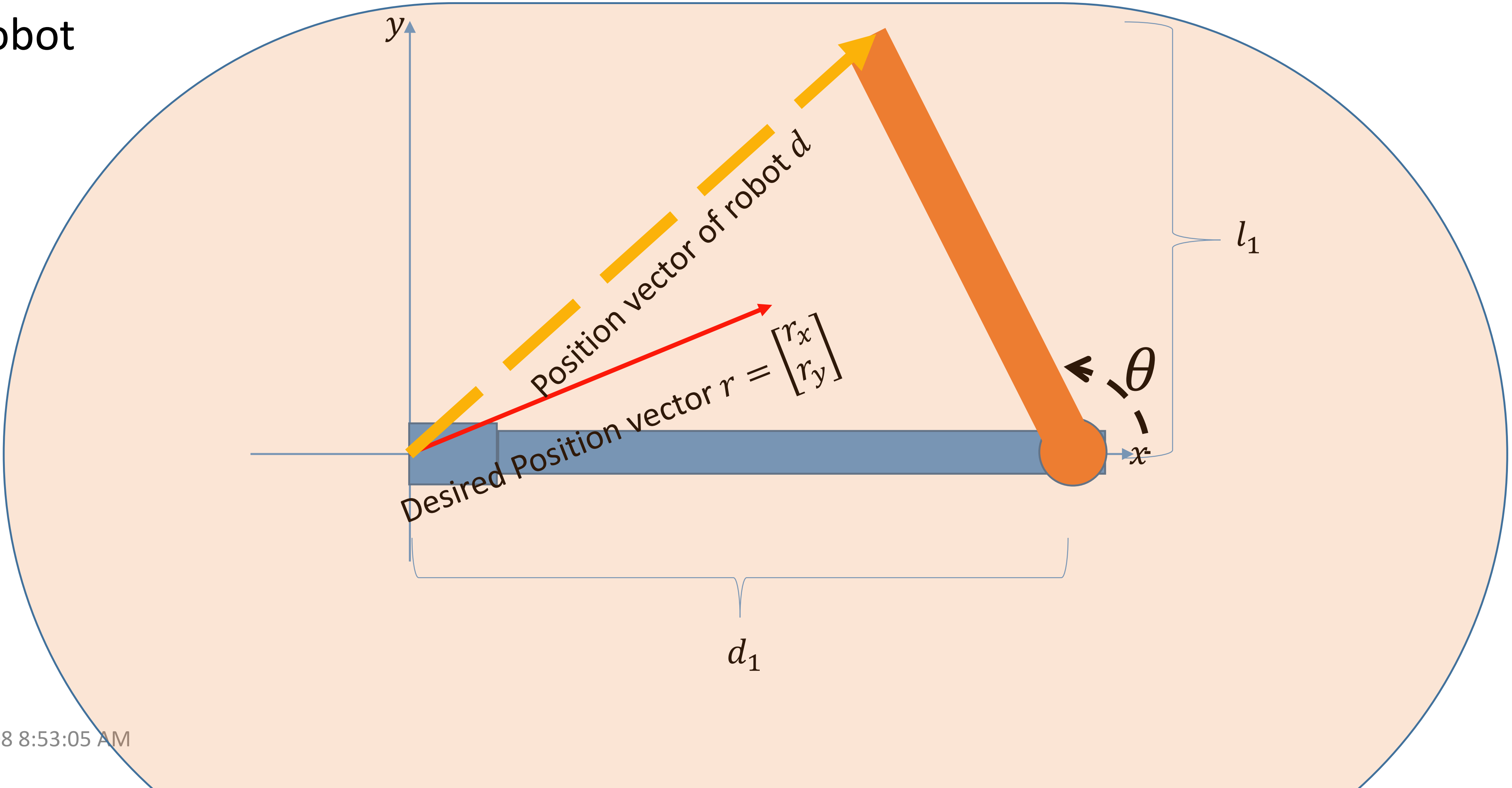
RP and PR Robots

- Kinematics
  - RP Robot
- Forward
  - $\begin{bmatrix} r_x \\ r_y \end{bmatrix} = \begin{bmatrix} d_1 cos\theta_1 \\ d_1 sin\theta_1 \end{bmatrix}$
- Inverse
  - $\theta_1 = atan2(r_x, r_y)$
  - $d_1 = \sqrt{r_x^2 + r_y^2}$



RP and PR Robots

- Kinematics
  - PR Robot



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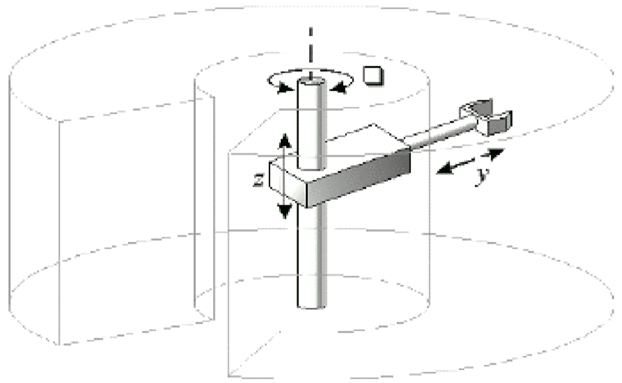
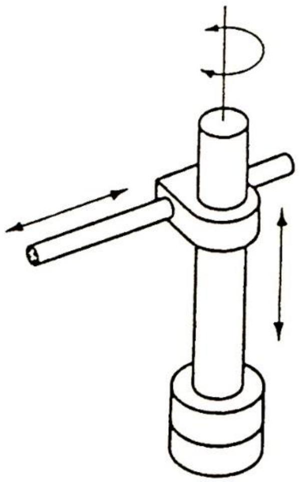
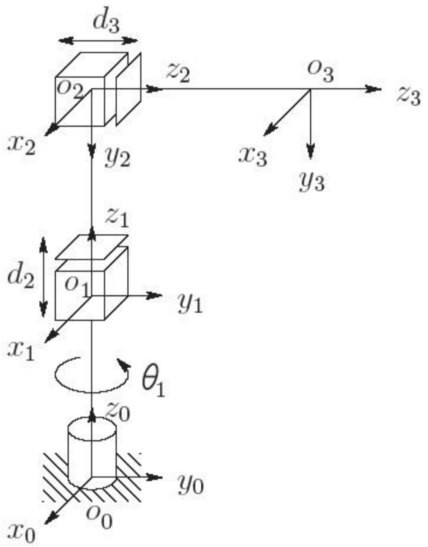


three-link cylindrical robot

- 3DOF: need to assign four coordinate frames
  1. Choose z_0 axis (axis of rotation for joint 1, base frame)
  2. Choose z_1 axis (axis of translation for joint 2)
  3. Choose z_2 axis (axis of translation for joint 3)
  4. Choose z_3 axis (tool frame)
     - This is again arbitrary for this case since we have described no wrist/gripper
     - Instead, define z_3 as parallel to z_2





- a_i: $\overline{O_i O_{i-1}}$ @ x_i
- α_i: $\angle(z_{i-1} \rightarrow z_i)$ @ x_i
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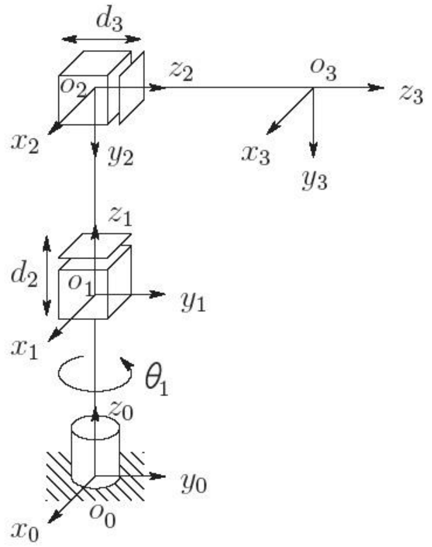
- θ_i: $\angle(x_{i-1} \rightarrow x_i)$ @ z_{i-1}

three-link cylindrical robot

- DH parameters
  - First, define the constant parameters a_i, α_i
  - Second, define the variable parameters θ_i, d_i

link	a_i	α_i	d_i	θ_i
1				
2				
3				



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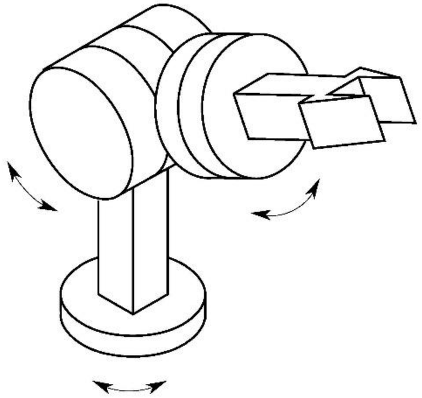
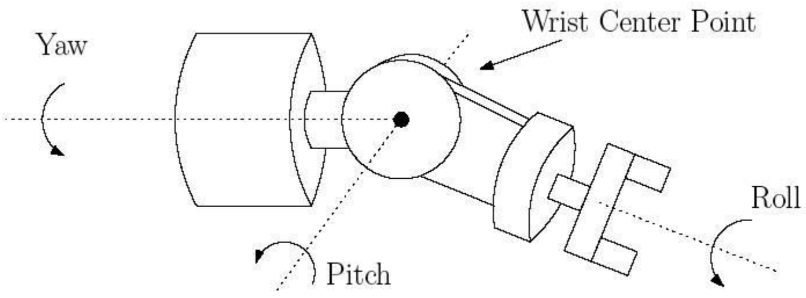
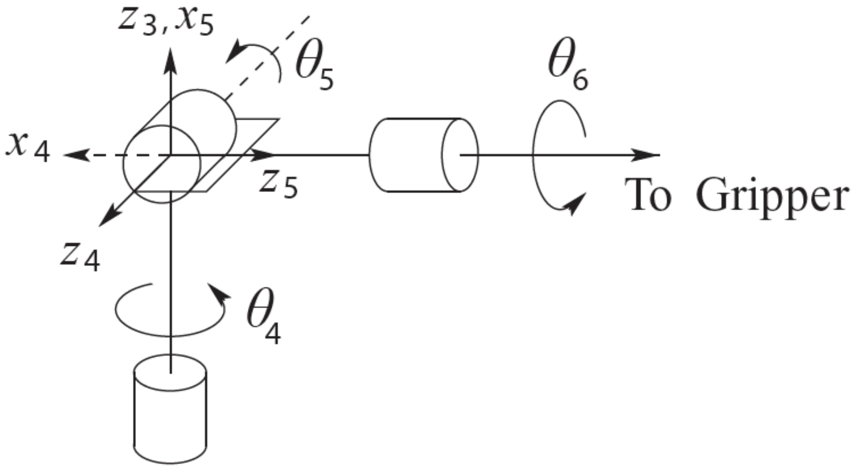
- θ_i: $\angle(x_{i-1} \to x_i)$ @ z_{i-1}

spherical wrist



- 3DOF: need to assign four coordinate frames
  - yaw, pitch, roll (θ_4, θ_5, θ_6) all intersecting at one point *o* (wrist center)
  1. Choose z_3 axis (axis of rotation for joint 4)
  2. Choose z_4 axis (axis of rotation for joint 5)
  3. Choose z_5 axis (axis of rotation for joint 6)
  4. Choose tool frame:
     - z_6 (*a*) is collinear with z_5
     - y_6 (*s*) is in the direction the gripper closes
     - x_6 (*n*) is chosen with a right-handed convention





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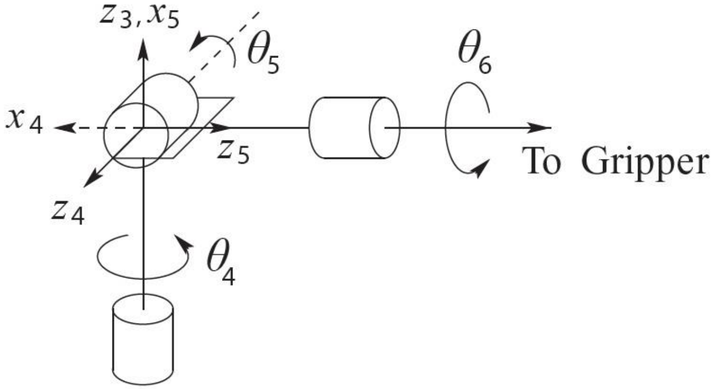
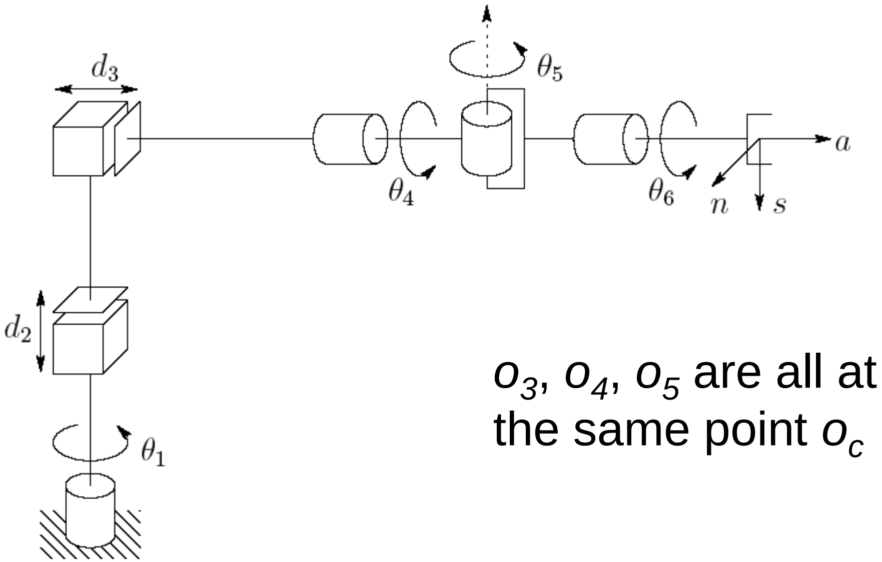
- DH parameters
  - First, define the constant parameters a_i, α_i
  - Second, define the variable parameters θ_i, d_i

link	a_i	α_i	d_i	θ_i



cylindrical robot with spherical wrist

- 6DOF: need to assign seven coordinate frames
  - But we already did this for the previous two examples, so we can fill in the table of DH parameters:

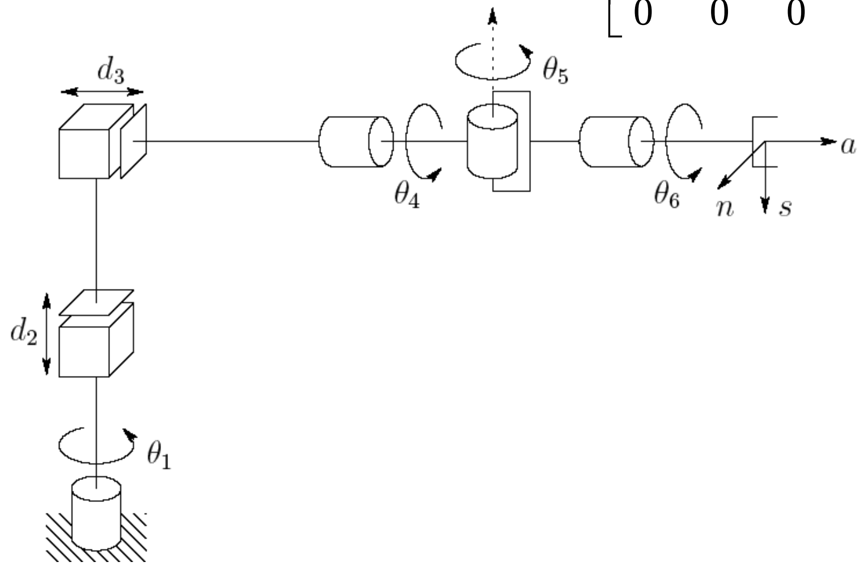


link	a_i	α_i	d_i	θ_i
1				
2				
3				
4				
5				
6				

o_3, o_4, o_5 are all at the same point o_c

- Note that z_3 (axis for joint 4) is collinear with z_2 (axis for joint 3), thus we can make the following combination:

$$ {}^0_6H = {}^0_3H\,{}^3_6H = \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} $$



$$ \begin{cases} r_{11} = c_1c_4c_5c_6 - c_1s_4s_6 + s_1s_5c_6 \\ r_{21} = s_1c_4c_5c_6 - s_1s_4s_6 - c_1s_5c_6 \\ r_{31} = -s_4c_5c_6 - c_4s_6 \\ r_{12} = -c_1c_4c_5s_6 - c_1s_4c_6 - s_1s_5c_6 \\ r_{22} = -s_1c_4c_5s_6 - s_1s_4s_6 + c_1s_5c_6 \\ r_{32} = s_4c_5c_6 - c_4c_6 \\ r_{13} = c_1c_4s_5 - s_1c_5 \\ r_{23} = s_1c_4s_5 + c_1c_5 \\ r_{33} = -s_4s_5 \\ d_x = c_1c_4s_5d_6 - s_1c_5d_6 - s_1d_3 \\ d_y = s_1c_4s_5d_6 + c_1c_5d_6 + c_1d_3 \\ d_z = -s_4s_5d_6 + d_1 + d_2 \end{cases} $$

- a_i: $\overline{O_iO_{i-1}}$ @ x_i
- α_i: $\angle(z_{i-1} \to z_i)$ @ x_i
- d_i: $\overline{O_iO_{i-1}}$ @ z_{i-1}
- θ_i: $\angle(x_{i-1} \to x_i)$ @ z_{i-1}

the Stanford manipulator

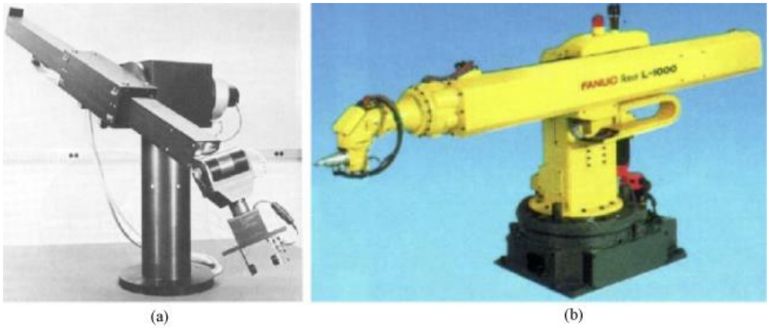
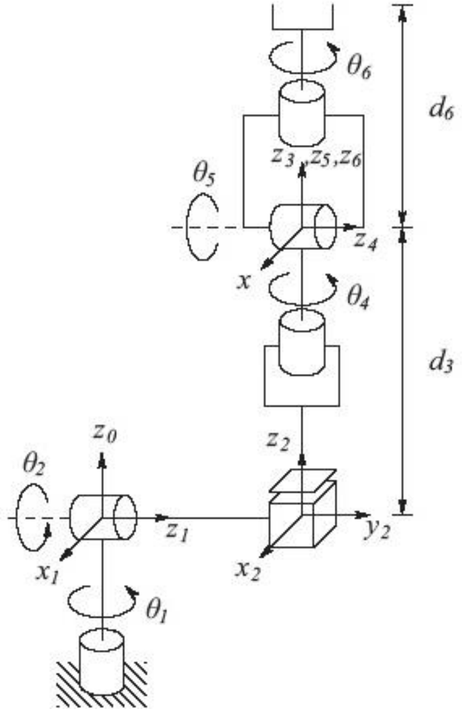


Figure 1: Stanford manipulator.

- 6DOF: need to assign seven coordinate frames:
  1. Choose z_0 axis (axis of rotation for joint 1, base frame)
  2. Choose z_1-z_5 axes (axes of rotation/translation for joints 2-6)
  3. Choose x_i axes
  4. Choose tool frame
  5. Fill in table of DH parameters:

link	a_i	α_i	d_i	θ_i
1				
2				
3				
4				
5				
6				



- the individual homogeneous transformations:

$$
{}^0_1H = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ {}^1_2H = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ {}^2_3H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^3_4H = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ {}^4_5H = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ {}^5_6H = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

- Finally, combine to give the complete description of the forward kinematics:

$$ {}^{0}_{6}H = {}^{0}_{1}H \cdots {}^{5}_{6}H = \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ \begin{cases} r_{11} = c_1\left[c_2\left(c_4c_5c_6 - s_4s_6\right) - s_2s_5c_6\right] - d_2\left(s_4c_5c_6 + c_4s_6\right) \\ r_{21} = s_1\left[c_2\left(c_4c_5c_6 - s_4s_6\right) - s_2s_5c_6\right] + c_1\left(s_4c_5c_6 + c_4s_6\right) \\ r_{31} = -s_2\left(c_4c_5c_6 - s_4s_6\right) - c_2s_5c_6 \\ r_{12} = c_1\left[-c_2\left(c_4c_5s_6 + s_4c_6\right) + s_2s_5s_6\right] - s_1\left(-s_4c_5s_6 + c_4c_6\right) \\ r_{22} = -s_1\left[-c_2\left(c_4c_5s_6 - s_4c_6\right) - s_2s_5s_6\right] + c_1\left(-s_4c_5s_6 + c_4s_6\right) \\ r_{32} = s_2\left(c_4c_5s_6 + s_4c_6\right) + c_2s_5s_6 \\ r_{13} = c_1\left(c_2c_4s_5 + s_2c_5\right) - s_1s_4s_5 \\ r_{23} = s_1\left(c_2c_4s_5 + s_2c_5\right) + c_1s_4s_5 \\ r_{33} = -s_2c_4s_5 + c_2c_5 \\ d_x = c_1s_2d_3 - s_1d_2 + d_6\left(c_1c_2c_4s_5 + c_1c_5s_2 - s_1s_4s_5\right) \\ d_y = s_1s_2d_3 + c_1d_2 + d_6\left(c_1s_4s_5 + c_2c_4s_1s_5 + c_5s_1s_2\right) \\ d_z = c_2d_3 + d_6\left(c_2c_5 - c_4s_2s_5\right) \end{cases} $$

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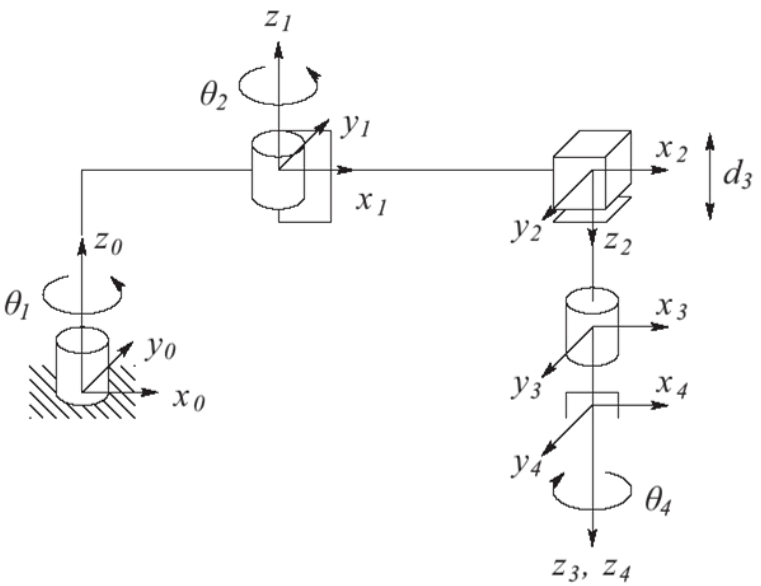
- θ_i: $\angle(x_{i-1} \rightarrow x_i)$ @ z_{i-1}

the SCARA manipulator

- 4DOF: need to assign five coordinate frames:
  1. Choose z_0 axis (axis of rotation for joint 1, base frame)
  2. Choose z_1-z_3 axes (axes of rotation/translation for joints 2-4)
  3. Choose x_i axes
  4. Choose tool frame
  5. Fill in table of DH parameters:

link	a_i	α_i	d_i	θ_i
1				
2				
3				
4				



Suggested insertion: photo of SCARA manipulator

- the individual homogeneous transformations:

$$
{}^0_1H = \begin{bmatrix} c_1 & -s_1 & 0 & a_1c_1 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ {}^1_2H = \begin{bmatrix} c_2 & s_2 & 0 & a_2c_2 \\ s_2 & -c_2 & 0 & a_2s_2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ {}^2_3H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ {}^3_4H = \begin{bmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^0_6H = {}^0_1H \cdots {}^5_6H = \begin{bmatrix} c_{12}c_4 + s_{12}s_4 & -c_{12}s_4 + s_{12}c_4 & 0 & a_1c_1 + a_2c_{12} \\ s_{12}c_4 - c_{12}s_4 & -s_{12}s_4 - c_{12}c_4 & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & -1 & -d_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$coordinate - i : \begin{cases} \theta_i & \text{revolute} \\ d_i & \text{prismatic} \end{cases}$

Joint coordinate-i: $\boxed{q_i = \bar{\varepsilon}_i \theta_i + \varepsilon_i d_i}$

with $\varepsilon_i = \begin{cases} 0 & \text{revolute} \\ 1 & \text{prismatic} \end{cases}$

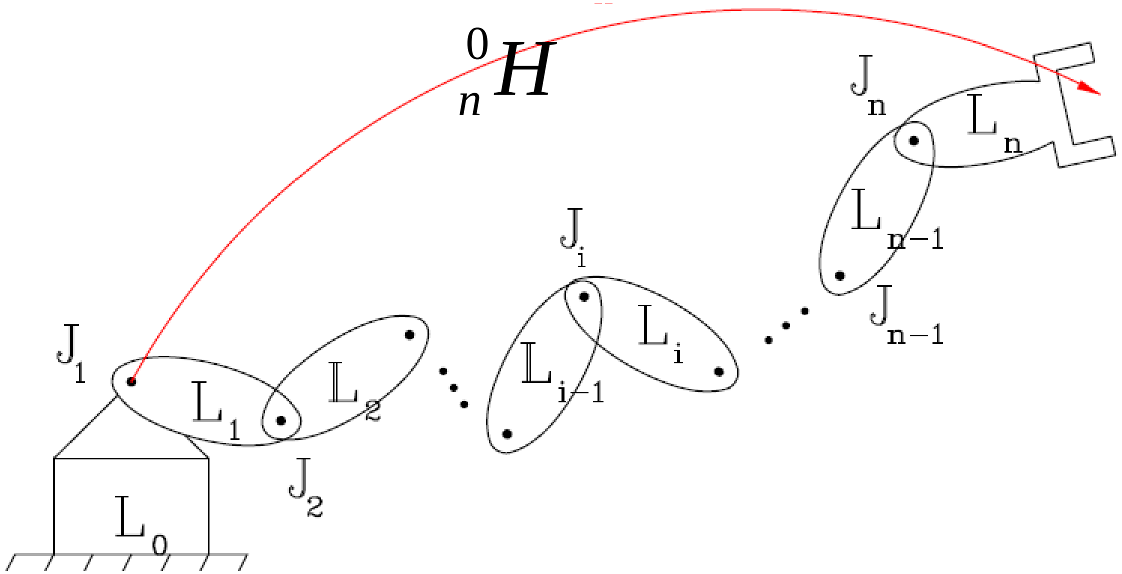
and $\bar{\varepsilon}_i = 1 - \varepsilon_i$

Joint Coordinate Vector: $\boxed{q = (q_1 q_2 \dots q_n)^T}$

Inverse Kinematics

Direct Kinematic Model:

- The direct kinematic model consists in a function $\mathbf{f}(\mathbf{q})$ mapping the joint position variables $\mathbf{q} \in \mathbb{R}^n$ to the position/orientation of the end effector.
- The definition of $\mathbf{f}(\mathbf{q})$ is conceptually simple, and a general approach for its computation has been defined.



Inverse Kinematics

- Find the values of joint parameters that will put the tool frame at a desired position and orientation (within the workspace)
  - Given *H*:

$$H = \begin{bmatrix} R & o \\ 0 & 1 \end{bmatrix} \in SE(3) \qquad H = \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix} \in SE(3)$$

The Euclidean group for SE(3) is used for the kinematics of a rigid body, in classical mechanics

  - Find *all* solutions to: $^0_6H(q_1, \ldots, q_n) = H$

  - Noting that: $^0_6H(q_1, \ldots, q_n) = {}^0_1H(q_1) \cdots {}^{n-1}_{n}H(q_n)$

  - This gives 12 (nontrivial) equations with *n* unknowns

Inverse Kinematic Model:

- The inverse kinematics consists in finding a function $\mathbf{g}(\mathbf{x})$ mapping the position/orientation of the end-effector to the corresponding joint variables $\mathbf{q}$: **the problem is not simple!**

A general approach for the solution of this problem does not exist

On the other hand, for the most common kinematic structures, a scheme for obtaining the solution has been found. Unfortunately

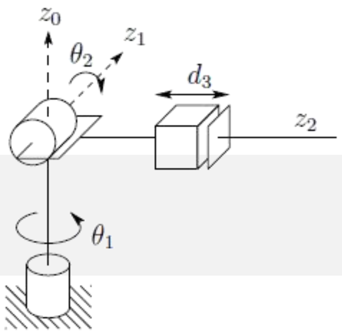
- The solution is not unique. In general we have:
  - No solution (e.g. starting with a position $\mathbf{x}$ not in the workspace);
  - A finite set of solutions (one or more);
  - Infinite solutions.
- We seek for closed form solutions, and not based on numerical techniques:
  - The analytic solution is more efficient from the computational point of view;
  - If the solutions are known analytically, it is possible to select one of them on the basis of proper criteria.

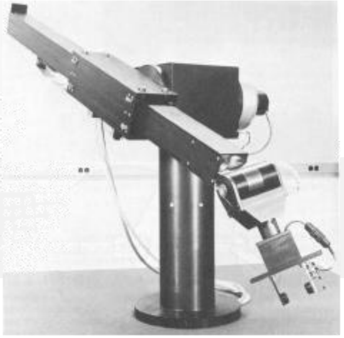
Closed-form solutions are ideal!

In order to obtain a closed form solution to the inverse kinematic problem, two approaches are possible:

- **An algebraic approach**, i.e. elaborations of the kinematic equations until a suitable set of (simple) equations is obtained for the solution

- **A geometric approach** based, when possible, on geometrical considerations, dependent on the kinematic structure of the manipulator and that may help in the solution.





Algebraic Approach

For a 6 dof manipulator, the kinematic model is described by the equation

$$ {}^{0}_{6}H(q_1,\dots,q_n) = {}^{0}_{1}H(q_1)\cdots{}^{n-1}_{n}H(q_n) $$

equivalent to 12 equations in the 6 unknowns q_i, $i = 1,\dots,6$.

Example: spherical manipulator (only 3 dof)

$$ H = \begin{bmatrix} 0.5868 & -0.6428 & 0.4394 & -0.4231 \\ 0.5265 & 0.7660 & 0.3687 & 0.9504 \\ -0.5736 & 0.0000 & 0.8192 & 0.4096 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C_1C_2 & -S_1 & C_1S_2 & -d_2S_1 + d_3C_1S_2 \\ C_2S_1 & C_1 & S_1S_2 & d_2C_1 + d_3S_1S_2 \\ -S_2 & 0 & C_2 & d_3C_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Since both the numerical values of ${}^{0}_{6}H$ and the structure of the ${}^{i-1}_{i}H$ matrices are known, by suitable pre- / post-multiplications it is possible to obtain

$$ [{}^{0}_{1}H(q_1)\cdots{}^{i-1}_{i}H(q_i)]^{-1}({}^{0}_{6}H)[{}^{j}_{j+1}H(q_{j+1})\cdots{}^{5}_{6}H(q_6)]^{-1} = {}^{i}_{i+1}H(q_1)\cdots{}^{j-1}_{j}H(q_j) \quad i < j $$

obtaining 12 new equations for each couple (i, j), $i < j$.

By selecting the most simple equations among all those obtained, it might be possible to obtain a solution to the problem.

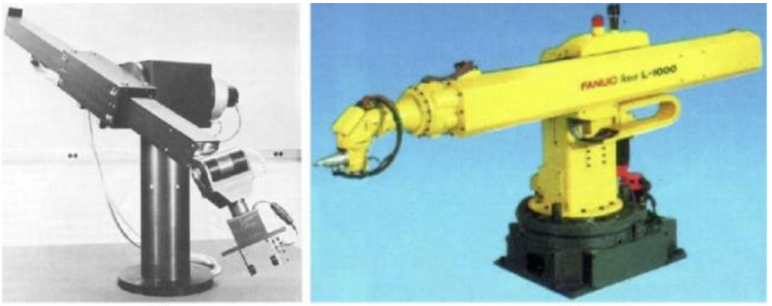

(a) (b)

Figure 1: Stanford manipulator.

the Stanford manipulator

- For a given H:

$$H = \begin{bmatrix} 0 & 1 & 0 & -0.154 \\ 0 & 0 & 1 & 0.763 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Find θ_1, θ_2, d_3, θ_4, θ_5, θ_6:

$$\underbrace{\begin{aligned}
c_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] - d_2(s_4c_5c_6 + c_4s_6) &= 0 \\
s_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] + c_1(s_4c_5c_6 + c_4s_6) &= 0 \\
-s_2(c_4c_5c_6 - s_4s_6) - c_2s_5c_6 &= 1 \\
c_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] - s_1(-s_4c_5s_6 + c_4c_6) &= 1 \\
-s_1[-c_2(c_4c_5s_6 - s_4c_6) - s_2s_5s_6] + c_1(-s_4c_5s_6 + c_4s_6) &= 0 \\
s_2(c_4c_5s_6 + s_4c_6) + c_2s_5s_6 &= 0 \\
c_1(c_2c_4s_5 + s_2c_5) - s_1s_4s_5 &= 0 \\
s_1(c_2c_4s_5 + s_2c_5) + c_1s_4s_5 &= 1 \\
-s_2c_4s_5 + c_2c_5 &= 0 \\
c_1s_2d_3 - s_1d_2 + d_6(c_1c_2c_4s_5 + c_1c_5s_2 - s_1s_4s_5) &= -0.154 \\
s_1s_2d_3 + c_1d_2 + d_6(c_1s_4s_5 + c_2c_4s_1s_5 + c_5s_1s_2) &= 0.763 \\
c_2d_3 + d_6(c_2c_5 - c_4s_2s_5) &= 0
\end{aligned}}$$

- One solution: $\theta_1 = \pi/2$, $\theta_2 = \pi/2$, $d_3 = 0.5$, $\theta_4 = \pi/2$, $\theta_5 = 0$, $\theta_6 = \pi/2$

Geometric Approach

General considerations that may help in finding solutions with algebraic techniques do not exist, being these strictly related to the mathematical expression of the direct kinematic model. On the other hand, it is often possible to exploit considerations related to the *geometrical structure* of the manipulator.

PIEPER APPROACH

Many industrial manipulators have a **kinematically decoupled structure**, for which it is possible to decompose the problem in two (simpler) sub-problems:

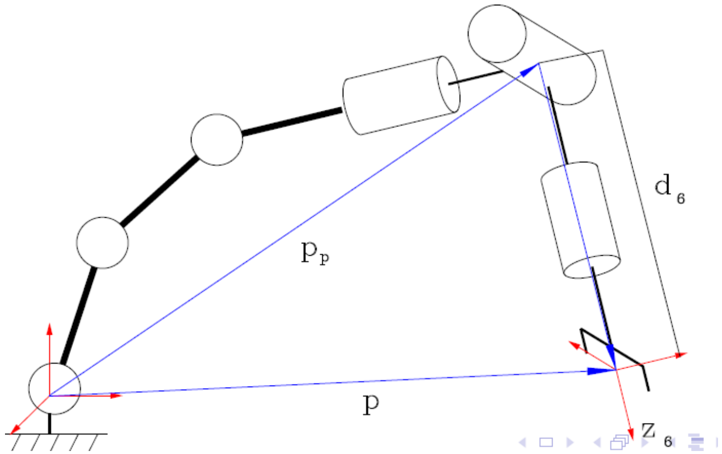
1. Inverse kinematics for the position $(x, y, z) \rightarrow q_1, q_2, q_3$
2. Inverse kinematics for the orientation $\mathbf{R} \rightarrow q_4, q_5, q_6.$

PIEPER APPROACH: *Given a 6 dof manipulator, sufficient condition to find a closed form solution for the IK problem is that the kinematic structure presents:*

- *three consecutive rotational joints with* axes intersecting in a single point

or

- *three consecutive rotational joints with* parallel axes.



In many 6 dof industrial manipulators, the first 3 dof are usually devoted to *position* the wrist, that has 3 additional dof give the correct *orientation* to the end-effector.

In these cases, it is quite simple to decompose the IK problem in the two sub-problems (position and orientation).

In case of a manipulator with a spherical wrist, a natural choice is to decompose the problem in

1. IK for the point $\mathbf{p}_p$ (center of the spherical wrist)
2. solution of the orientation IK problem

Therefore:

1. The point $\mathbf{p}_p$ is computed since ${}^0_6H$ is known (submatrices $\mathbf{R}$ and $\mathbf{p}$):

$$\mathbf{p}_p = \mathbf{p} - d_6\mathbf{a}$$

   $\mathbf{p}_p$ depends only on the joint variables q_1, q_2, q_3;
2. The IK problem is solved for q_1, q_2, q_3;
3. The rotation matrix ${}^0\mathbf{R}_3$ related to the first three joints is computed;
4. The matrix ${}^3\mathbf{R}_6 = {}^0\mathbf{R}_3^T \ \mathbf{R}$ is computed;
5. The IK problem for the rotational part is solved (Euler).

kinematic decoupling

- Appropriate for systems that have an arm a wrist
  – Such that the wrist joint axes are aligned at a point
- For such systems, we can split the inverse kinematics problem into two parts:
  1. Inverse position kinematics: position of the wrist center
  2. Inverse orientation kinematics: orientation of the wrist
- First, assume 6DOF, the last three intersecting at o_c

$$
{}_6^0R(q_1,\ldots,q_6) = R
$$
$$
{}_6^0o(q_1,\ldots,q_6) = o
$$

- Use the position of the wrist center to determine the first three joint angles…

- Now, origin of tool frame, o_6, is a distance d_6 translated along z_5 (since z_5 and z_6 are collinear)
  - Thus, the third column of R is the direction of z_6 (w/ respect to the base frame) and we can write:

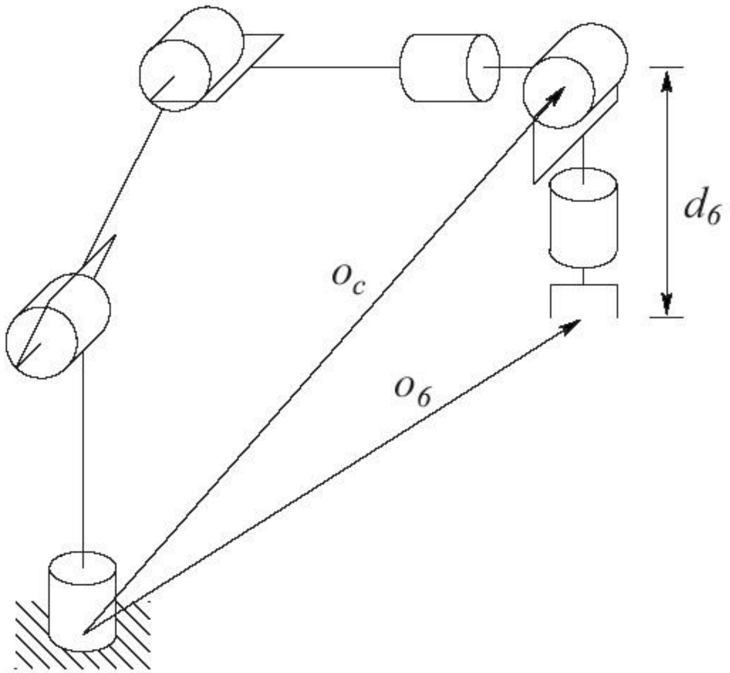
$$o = o_6^0 = o_c^o + d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

  - Rearranging:

$$o_c^o = o - d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

  - Calling $o = [o_x\ o_y\ o_z]^T$, $o_c^0 = [x_c\ y_c\ z_c]^T$

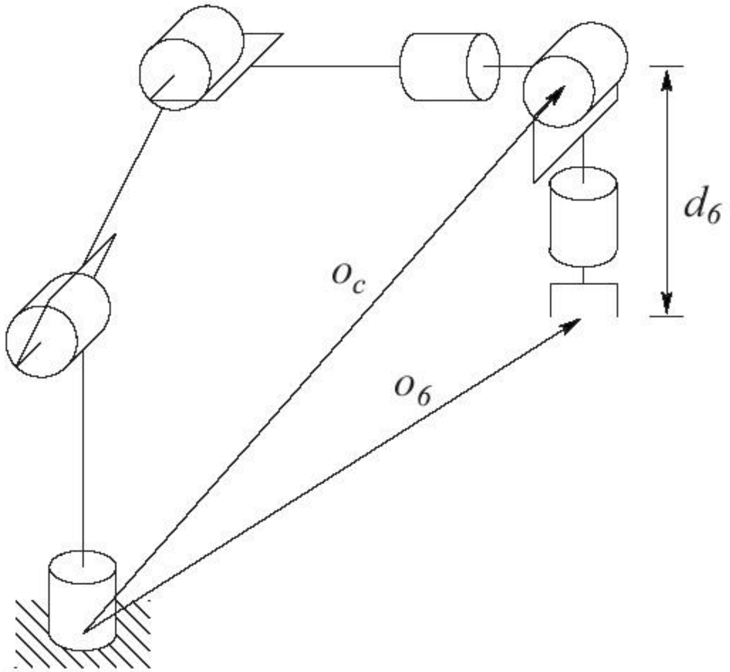
$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} o_x - d_6 r_{13} \\ o_y - d_6 r_{23} \\ o_z - d_6 r_{33} \end{bmatrix}$$



- Since $[x_c\ y_c\ z_c]^T$ are determined from the first three joint angles, our forward kinematics expression now allows us to solve for the first three joint angles decoupled from the final three.
  - Thus we now have R_3^0
  - Note that: $$R = {}^0_3R\,{}^3_6R$$
  - To solve for the final three joint angles:

$$ {}^3_6R = \left({}^0_3R\right)^{-1} R = \left({}^0_3R\right)^T R $$

  - Since the last three joints for a spherical wrist, we can use a set of Euler angles to solve for them



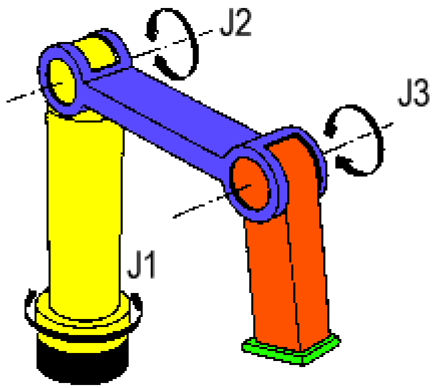
Inverse position

- Now that we have $[x_c\ y_c\ z_c]^T$ we need to find q_1, q_2, q_3
  - Solve for q_i by projecting onto the x_{i-1}, y_{i-1} plane, solve trig problem
  - Two examples: elbow (RRR) and spherical (RRP) manipulators
  - For example, for an elbow manipulator, to solve for θ_1, project the arm onto the x_0, y_0 plane

Background: two argument atan

- We use atan2(·) instead of atan(·) to account for the full range of angular solutions
  - Called ‘four-quadrant’ arctan

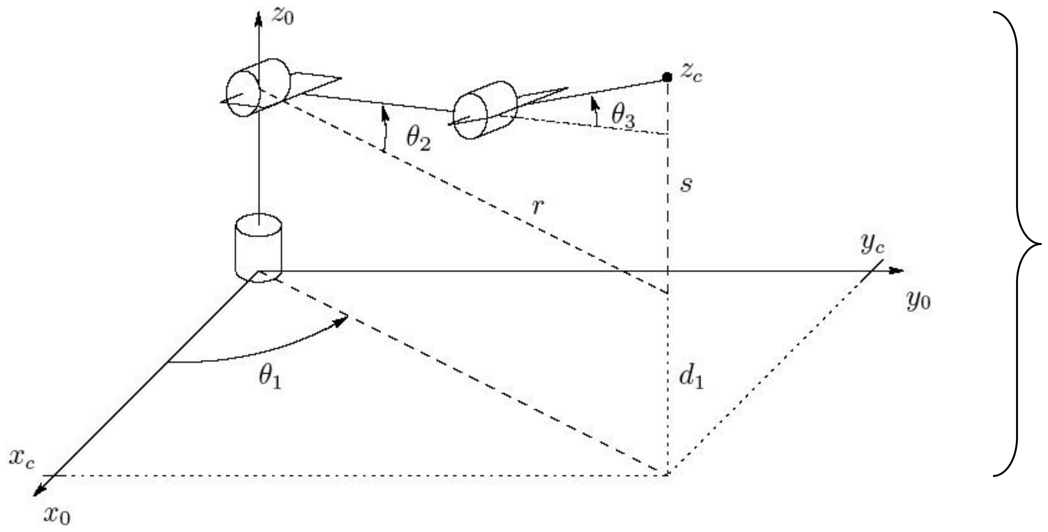
$$\mathbf{atan}2(y,x)=\begin{cases}-\mathbf{atan}2(-y,x) & y<0\\ \pi-\mathbf{atan}\left(-\frac{y}{x}\right) & y\geq 0, x<0\\ \mathbf{atan}\left(\frac{y}{x}\right) & y\geq 0, x\geq 0\\ \frac{\pi}{2} & y>0, x=0\\ \textbf{undefined} & y=0, x=0\end{cases}$$

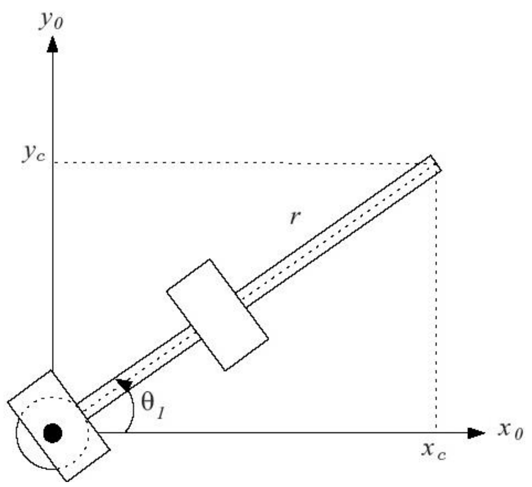


RRR manipulator
Anthropomorphic structure

1. To solve for θ_1, project the arm onto the x_0, y_0 plane

$$\theta_1 = \mathbf{atan2}(x_c, y_c)$$

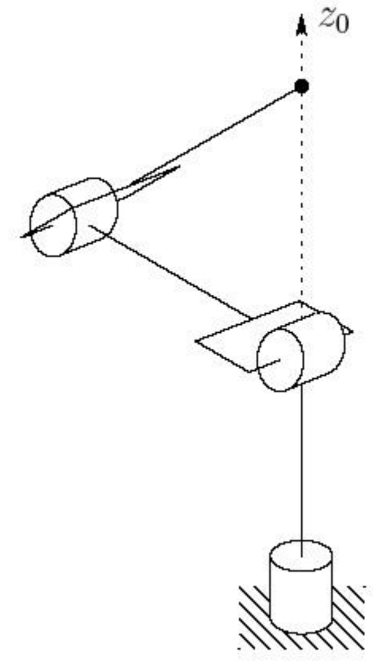




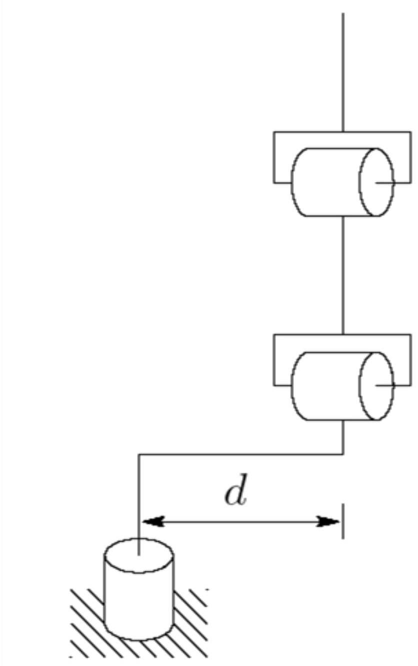
– Can also have: $\theta_1 = \pi + \mathbf{atan2}(x_c, y_c)$
  - This will of course change the solutions for θ_2 and θ_3

singular configurations, offsets

- If x_c=y_c=0, θ_1 is undefined
  - i.e. any value of θ_1 will work



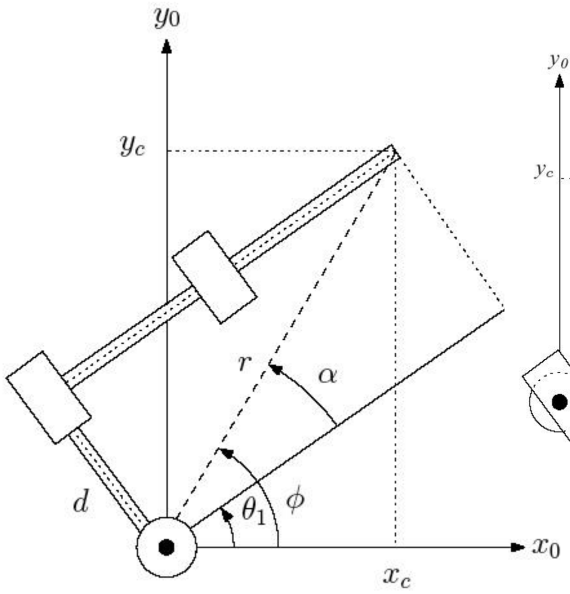
- If there is an offset, then we will have two solutions for θ_1: *left arm* and *right arm*
  - However, wrist centers cannot intersect z_0

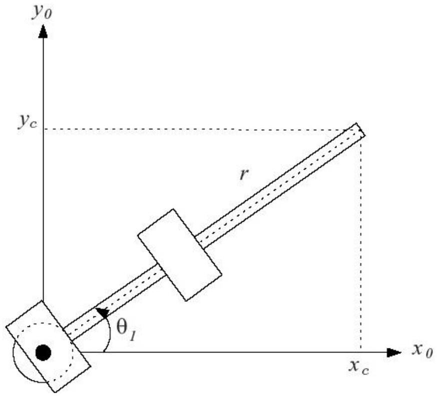


Left arm and right arm solutions

- Left arm:

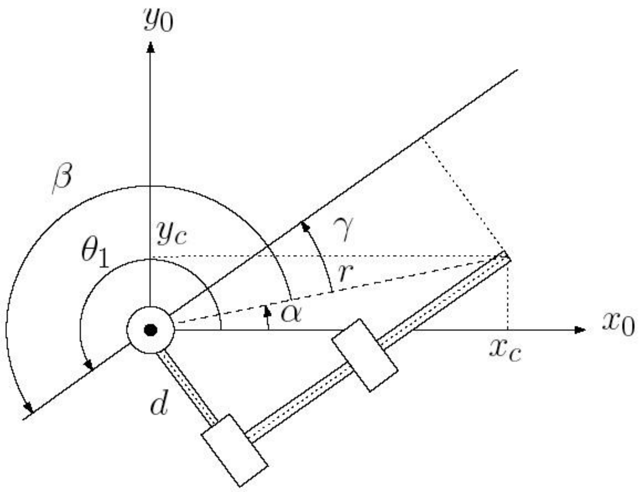
$$\theta_1 = \phi - \alpha$$
$$\phi = \textbf{atan2}(x_c, y_c)$$
$$\alpha = \textbf{atan2}\left(\sqrt{x_c^{\,2} + y_c^{\,2} - d^2}, d\right)$$





- Right arm:

$$\theta_1 = \alpha + \beta$$
$$\alpha = \textbf{atan2}(x_c, y_c)$$
$$\beta = \pi + \textbf{atan2}\left(\sqrt{x_c^{\,2} + y_c^{\,2} - d^2}, d\right)$$
$$= \textbf{atan2}\left(-\sqrt{x_c^{\,2} + y_c^{\,2} - d^2}, -d\right)$$



Left arm and right arm solutions

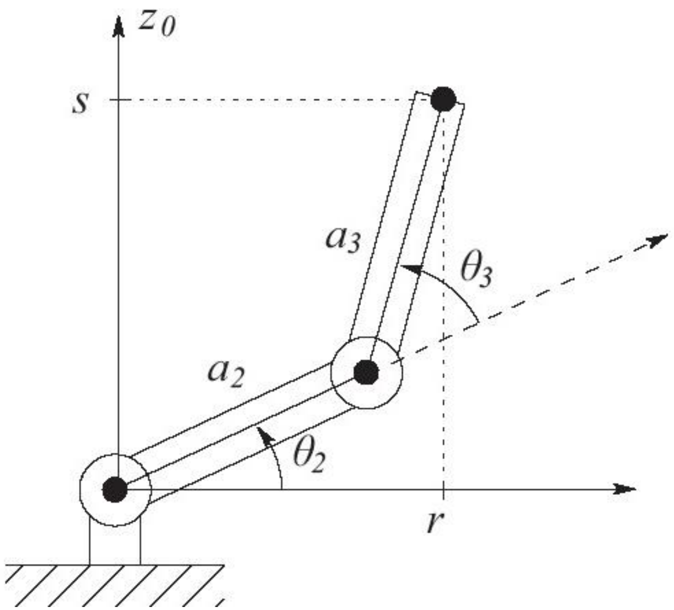
- Therefore there are in general two solutions for θ_1
- Finding θ_2 and θ_3 is identical to the planar two-link manipulator we have seen previously:

$$\cos\theta_3 = \frac{r^2 + s^2 - a_2^2 - a_3^2}{2a_2a_3}$$

$$r^2 = x_c^2 + y_c^2 - d^2$$

$$s = z_c - d_1$$

$$\Rightarrow \cos\theta_3 = \frac{x_c^2 + y_c^2 - d^2 + (z_c - d_1)^2 - a_2^2 - a_3^2}{2a_2a_3} \equiv D$$



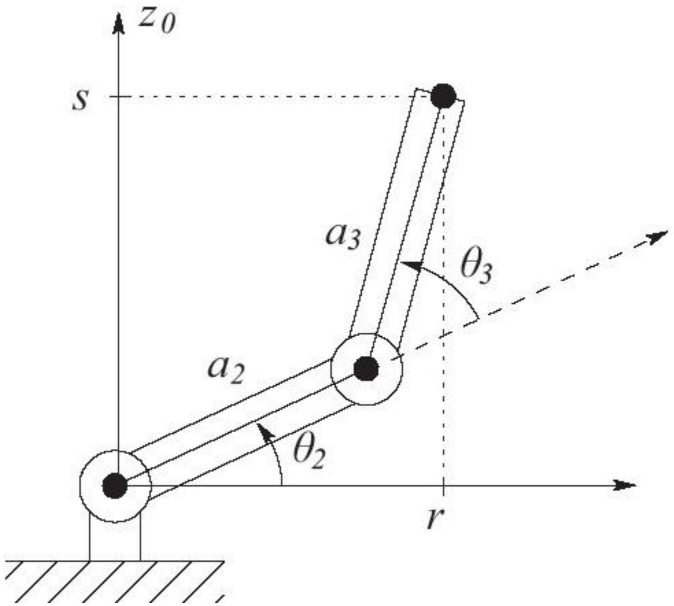
- Therefore we can find two solutions for θ_3

$$\theta_3 = \mathbf{atan2}\left(D, \pm\sqrt{1 - D^2}\right)$$

Left arm and right arm solutions

- The two solutions for θ_3 correspond to the elbow-down and elbow-up positions respectively
- Now solve for θ_2:

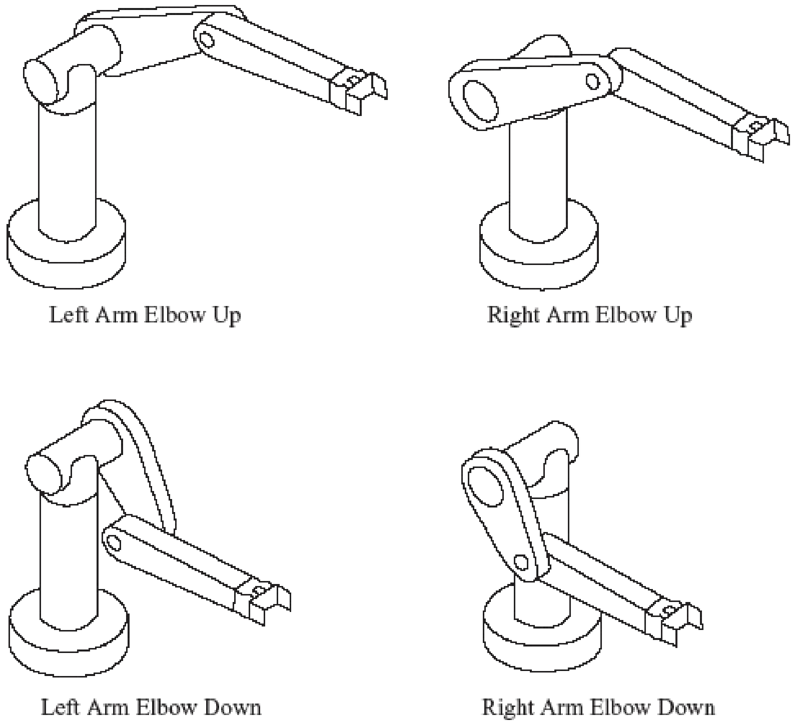
$$\begin{aligned}\theta_2 &= \mathbf{atan}2(r,s) - \mathbf{atan}2(a_2 + a_3c_3, a_3s_3) \\ &= \mathbf{atan}2\left(\sqrt{x_c^{\,2} + y_c^{\,2} - d^2}, z_c - d_1\right) - \mathbf{atan}2(a_2 + a_3c_3, a_3s_3)\end{aligned}$$



- Thus there are two solutions for the pair (θ_2, θ_3)

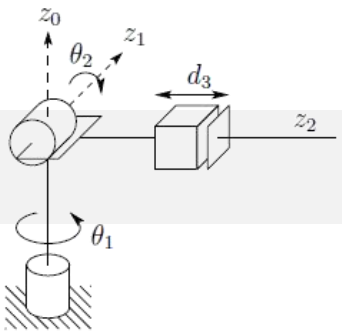
RRR: Four total solutions

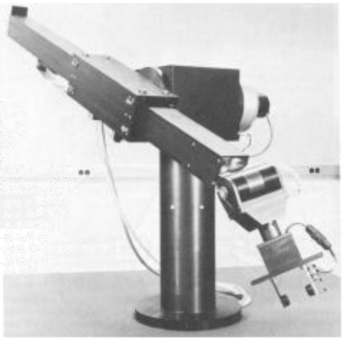
- In general, there will be a maximum of four solutions to the inverse *position* kinematics of an elbow manipulator
  - Ex: PUMA

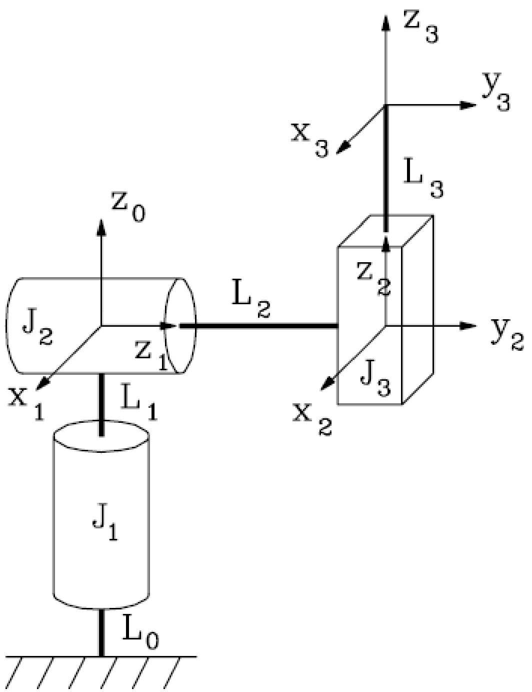


RRP manipulator

Solution of the spherical manipulator







Direct kinematic model:

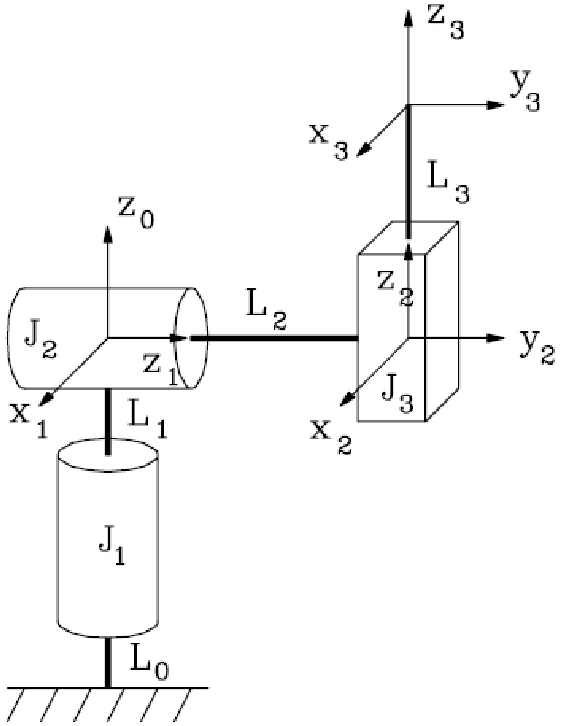
$$
{}^0_3H = \begin{bmatrix} \mathbf{n} & \mathbf{s} & \mathbf{a} & \mathbf{p} \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} C_1C_2 & -S_1 & C_1S_2 & -d_2S_1 + d_3C_1S_2 \\ C_2S_1 & C_1 & S_1S_2 & d_2C_1 + d_3S_1S_2 \\ -S_2 & 0 & C_2 & d_3C_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

If the whole matrix ${}^0_3H$ is known, it is possible to compute:

$$
\begin{cases} \theta_1 = \operatorname{atan2}(-s_x, s_y) \\ \theta_2 = \operatorname{atan2}(-n_z, a_z) \\ d_3 = p_z / \cos\theta_2 \end{cases}
$$

$\Longrightarrow$ Unfortunately, according to the Pieper approach only **p** is known!



We known only the position vector **p**.

From $\quad {}^0_3H = {}^0_1H\,{}^1_2H\,{}^2_3H \quad$ we have

$$({}^0_1H)^{-1}\,{}^0_3H = \begin{bmatrix} C_1 & S_1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -S_1 & C_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_z & s_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C_2 & 0 & S_2 & d_3 S_2 \\ S_2 & 0 & -C_2 & -d_3 C_2 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= {}^1_2H\,{}^2_3H$$

By equating the position vectors,

$$ {}^1\mathbf{p}_p = \begin{bmatrix} p_x C_1 + p_y S_1 \\ -p_z \\ -p_x S_1 + p_y C_1 \end{bmatrix} = \begin{bmatrix} d_3 S_2 \\ -d_3 C_2 \\ d_2 \end{bmatrix} $$

The vector ${}^1\mathbf{p}_p$ depends only on θ_2 and d_3! Let's define $a = \tan\theta_1/2$, then

$$ C_1 = \frac{1-a^2}{1+a^2} \qquad\qquad S_1 = \frac{2a}{1+a^2} $$

By substitution in the last element of ${}^1\mathbf{p}_p$

$$ (d_2 + p_y)a^2 + 2p_x a + d_2 - p_y = 0 \qquad \Longrightarrow \qquad a = \frac{-p_x \pm \sqrt{p_x^2 + p_y^2 - d_2^2}}{d_2 + p_y} $$

Two possible solutions! $((p_x^2 + p_y^2 - d_2^2) > 0??)$

Then

$$ \theta_1 = 2\ \mathrm{atan2}\left(-p_x \pm \sqrt{p_x^2 + p_y^2 - d_2^2},\ \ d_2 + p_y\right) $$

Vector ${}^1\mathbf{p}_p$ is defined as

$$ {}^1\mathbf{p}_p = \begin{bmatrix} p_x C_1 + p_y S_1 \\ -p_z \\ -p_x S_1 + p_y C_1 \end{bmatrix} = \begin{bmatrix} d_3 S_2 \\ -d_3 C_2 \\ d_2 \end{bmatrix} $$

From the first two elements

$$ \frac{p_x C_1 + p_y S_1}{-p_z} = \frac{d_3 S_2}{-d_3 C_2} $$

from which

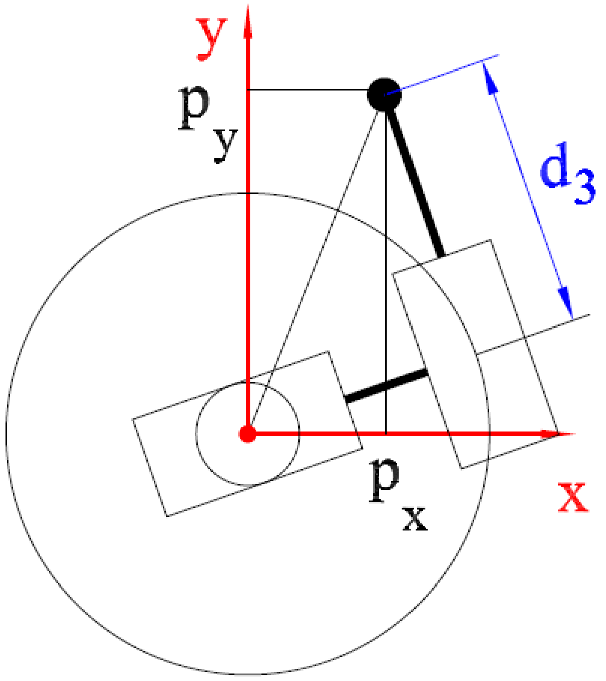
$$ \theta_2 = \operatorname{atan2}\left(p_x C_1 + p_y S_1,\ p_z\right) $$

Finally, if the first two elements are squared and added together

$$ d_3 = \sqrt{(p_x C_1 + p_y S_1)^2 + p_z^2} $$

Note that two possible solutions exist considering the position of the end-effector (wrist) only. If also the orientation is considered, the solution (if exists) is unique.

We have seen that the relation $(p_x^2 + p_y^2 - d_2^2) > 0$ must hold:



Numerical example: Given a spherical manipulator with $d_2 = 0.8\ m$ in the pose

$$\theta_1 = 20^o, \qquad \theta_2 = 30^o, \qquad d_3 = 0.5\ m$$

we have

$${}^0_3H = \left[\begin{array}{ccc|c} 0.8138 & -0.342 & 0.4698 & -0.0387 \\ 0.2962 & 0.9397 & 0.171 & 0.8373 \\ -0.5 & 0 & 0.866 & 0.433 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

- The solution based on the whole matrix ${}^0_3H$ is: $\theta_1 = 20^o,\ \theta_2 = 30^o,\ d_3 = 0.5$.
- The solution based on the vector $\mathbf{p}$ gives:

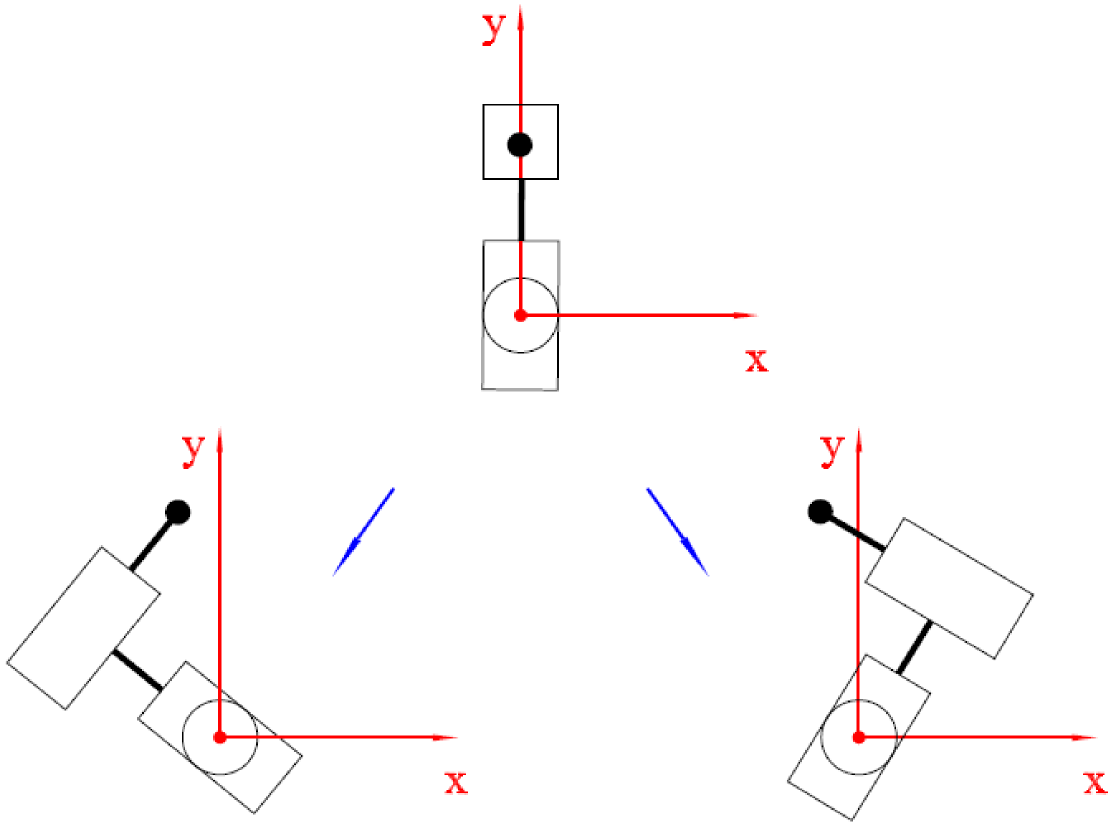
a) $\theta_1 = 20^o,\ \theta_2 = 30^o,\ d_3 = 0.5$ (with the "+" sign).

b) $\theta_1 = -14.7^o,\ \theta_2 = -30^o,\ d_3 = 0.5$ (with the "-" sign).

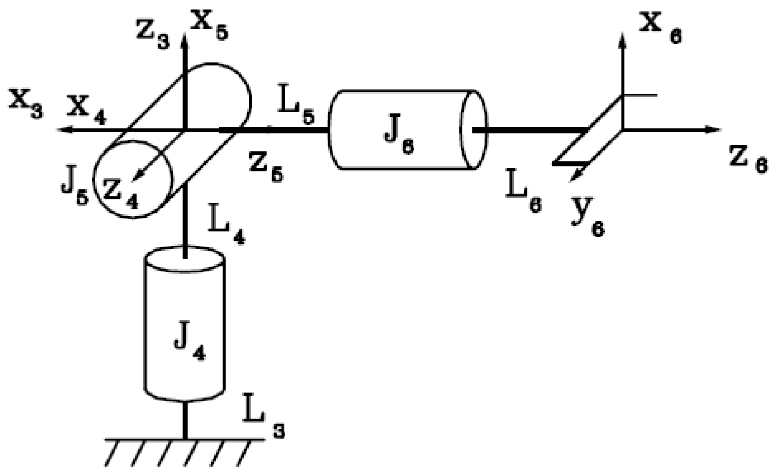
- The solution based on the vector **p** gives:

a) $\theta_1 = 20^o,\ \theta_2 = 30^o,\ d_3 = 0.5$ (with the "+" sign).

b) $\theta_1 = -14.7^o,\ \theta_2 = -30^o,\ d_3 = 0.5$ (with the "-" sign).



Solution of the spherical wrist



Let us consider the matrix

$$
{}^3\mathbf{R}_6 = \begin{bmatrix} n_x & s_x & a_x \\ n_y & s_y & a_y \\ n_z & s_z & a_z \end{bmatrix}
$$

From the direct kinematic equations one obtains

$$
{}^3\mathbf{R}_6 = \begin{bmatrix} C_4 C_5 C_6 - S_4 S_6 & -S_4 C_6 - C_4 C_5 S_6 & C_4 S_5 \\ S_4 C_5 C_6 + C_4 S_6 & C_4 C_6 - S_4 C_5 S_6 & S_4 S_5 \\ -S_5 C_6 & S_5 S_6 & C_5 \end{bmatrix}
$$

$$
{}^3\mathbf{R}_6 = \begin{bmatrix} C_4C_5C_6 - S_4S_6 & -S_4C_6 - C_4C_5S_6 & C_4S_5 \\ S_4C_5C_6 + C_4S_6 & C_4C_6 - S_4C_5S_6 & S_4S_5 \\ -S_5C_6 & S_5S_6 & C_5 \end{bmatrix}
$$

The solution is then computed as (ZYZ Euler angles):

- $\theta_5 \in [0, \pi]$:

$$
\begin{aligned}
\theta_4 &= \mathrm{atan2}\left(a_y,\ a_x\right) \\
\theta_5 &= \mathrm{atan2}\left(\sqrt{a_x^2 + a_y^2},\ \ a_z\right) \\
\theta_6 &= \mathrm{atan2}\left(s_z,\ -n_z\right)
\end{aligned}
$$

- $\theta_5 \in [-\pi, 0]$:

$$
\begin{aligned}
\theta_4 &= \mathrm{atan2}\left(-a_y,\ -a_x\right) \\
\theta_5 &= \mathrm{atan2}\left(-\sqrt{a_x^2 + a_y^2},\ \ a_z\right) \\
\theta_6 &= \mathrm{atan2}\left(-s_z,\ n_z\right)
\end{aligned}
$$

Numerical example: Let us consider a spherical wrist in the pose

$$\theta_4 = 10^o \qquad \theta_5 = 20^o, \qquad \theta_6 = 30^o$$

Then

$${}^3\mathbf{R}_6 = \begin{bmatrix} 0.7146 & -0.6131 & 0.3368 \\ 0.6337 & 0.7713 & 0.0594 \\ -0.2962 & 0.1710 & 0.9397 \end{bmatrix}$$

Therefore, if

- $\theta_5 \in [0,\ \pi]$ $\qquad \theta_4 = 10^o \qquad \theta_5 = 20^o, \qquad \theta_6 = 30^o$
- $\theta_5 \in [-\pi,\ 0]$ $\qquad \theta_4 = -170^o \qquad \theta_5 = -20^o, \qquad \theta_6 = -30^o$

Note that the PUMA has an anthropomorphic structure (4 solutions) and a spherical wrist (2 solutions):

$\Longrightarrow$ 8 different configurations are possible!