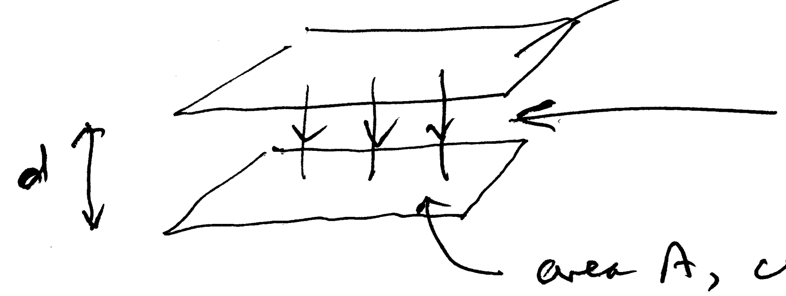
· of the electric field.

Example : parallel plate capacitor



area A, charge density $+\sigma$

$|\vec{E}| = \frac{\sigma}{\varepsilon_0}$ (constant).

area A, charge density $-\sigma$

- Energy density between plates

$$u = \frac{1}{2}\varepsilon_0 |\vec{E}|^2$$
$$= \frac{1}{2}\varepsilon_0 \frac{\sigma^2}{\varepsilon_0^2}$$
$$= \frac{1}{2\varepsilon_0}\sigma^2$$

$\Rightarrow$ Energy stored in electric field between

- plates

$$U = u \times \text{volume of field}$$
$$= \frac{1}{2}\frac{\sigma^2}{\varepsilon_0} A . d$$

Using the definition $C = \frac{Q}{\Delta V}$ of capacitance, and the fact that $Q = A\sigma$ and $\Delta V = E . d = \frac{\sigma}{\varepsilon_0} d$ $\left(V(\vec{r_2}) - V(\vec{r_1}) = -\int_{1\to 2} \vec{E}.d\vec{l} \right)$,

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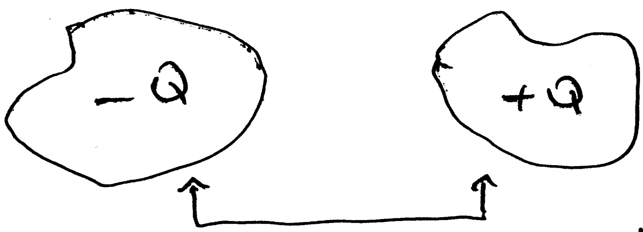
$$= \frac{1}{2} A\sigma . \frac{\sigma}{\varepsilon_0} d$$

$$= \frac{1}{2} \frac{\sigma^2}{\varepsilon_0} A . d \quad \checkmark \quad]$$

Another way to determine the energy stored in a capacitor

Consider a capacitor more generally:

charged conductor



potential difference ΔV (we have to do work against the $\vec{E}$ field to move a charge from $-Q$ to $+Q$.

By definition $\Delta V = \frac{1}{C} Q$

capacitance, depends on geometry of charge distributions and separation

We calculate the energy stored in the capacitor as the amount of work done to charge it.

Assume the conductors already have charge q, the potential difference is $\frac{q}{C}$ $\Rightarrow$ the amount of work to transfer an amount of charge dq from the negatively charged conductor to the positively charge conductor is

$$dW = \underbrace{\frac{q}{C}}_{\substack{\text{potential difference} \\ = \text{potential energy per unit charge}}} \cdot dq$$

$\Rightarrow$ total work in starting from zero charge and building up to Q is

$$W = \int dW = \frac{1}{C}\int_0^Q q\,dq$$

$$= \frac{1}{C}\left[\frac{1}{2}q^2\right]_0^Q$$

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ie. $U = \frac{1}{2}\frac{Q^2}{C}$.

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or as $U = \frac{1}{2} Q \Delta V$.

These apply for general capacitors (not just parallel plate).

INDUCTORS

In electrical circuits:

- resistors cause energy loss
- capacitors store energy in $\vec{E}$ fields
- inductors are circuit elements that store energy in $\vec{B}$ fields.

- Their operation is based on Faraday's law $\vec{\nabla} \times E(\vec{r}, t) = -\frac{\partial \vec{B}(\vec{r}, t)}{\partial t}$, which describes the effect of a changing magnetic field - it induces an emf in a circuit.

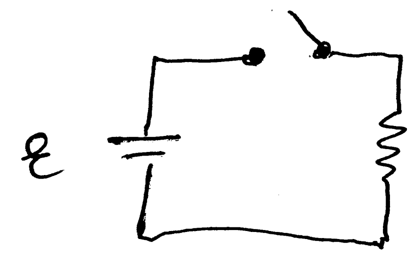
- Inductors are only active when currents are changing ($\Rightarrow$ magnetic fields) changing. They play a crucial role in controlling currents in circuits with time varying currents.

Self inductance

- A circuit carrying a changing current produces a changing magnetic field.

Faraday's law ⇒ an additional emf may be produced in the circuit

~~Ex~~ Example:



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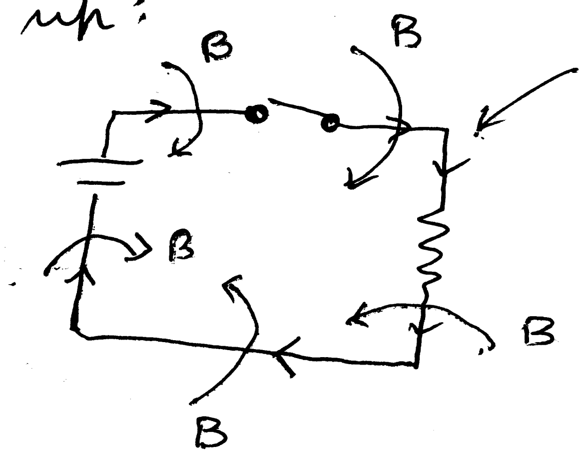

As field grows, flux through loop increases
⇒ induced emf.

<u>Lenz's law</u>: induced emf always <u>opposes</u> current increase and <u>slows down rate</u> of change of current.

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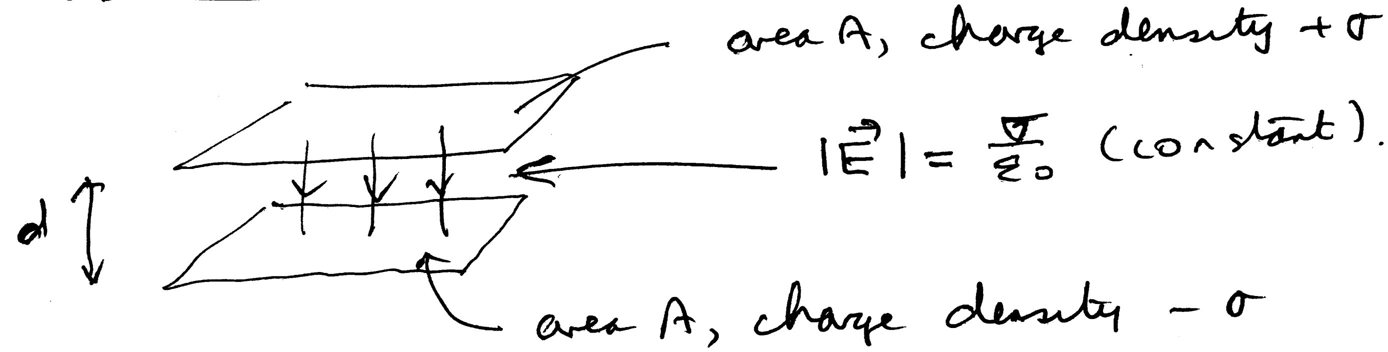

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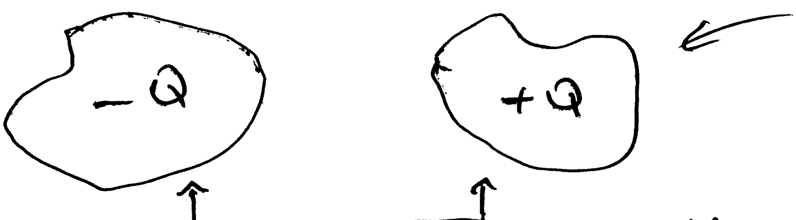

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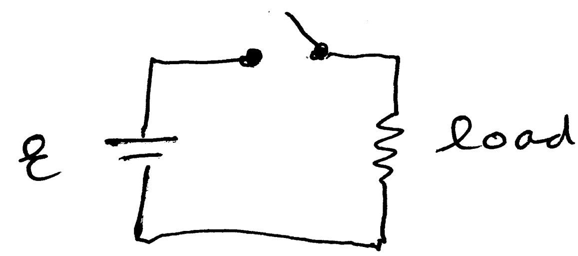

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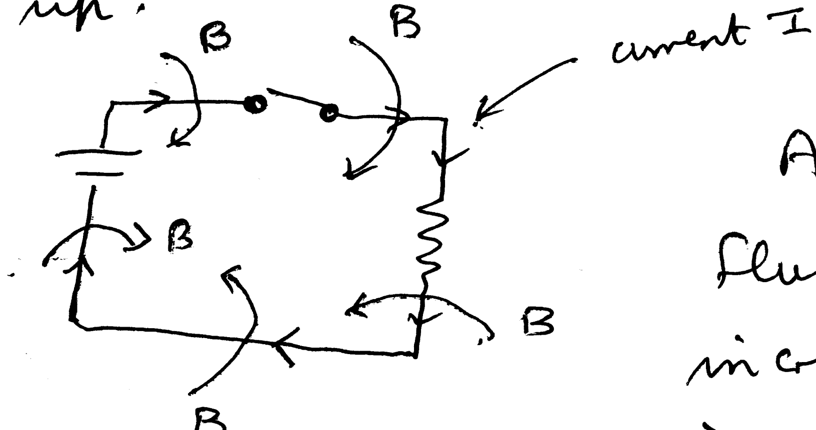


As field grows, flux through loop increases ⇒ induced emf.

<u>Lenz's law</u>: induced emf always <u>opposes</u> current increase and <u>slows down rate</u> of change of current.

- When a circuit or circuit element induces an emf in itself due to changing magnetic flux, we say there is self inductance (or just inductance). If we have two circuits in the vicinity of each other, the changing magnetic field of one can produce a changing magnetic field in the second and therefore an induced emf – in this case we talk of mutual inductance.
  e.g transformer

- Self inductance

The magnetic field set up in a circuit or circuit element is proportional to the current I ⇒ magnetic flux is proportional to I.

- The inductance L is defined by

$$\boxed{\Phi = L\, I}$$

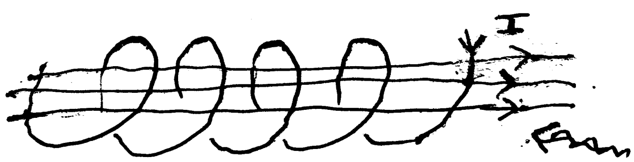
EFFECT: magnetic flux

CAUSE: current → magnetic field

inductance, determines the strength of the relationship between cause (I) and effect Φ

- L will depend on the geometry of the circuit or circuit element.

Example: solenoid with n turns of wire per unit length, and carrying current I.



constant magnetic field

$$|\vec{B}| = \frac{nI}{\varepsilon_0 c^2}$$

If A = cross-sectional area of solenoid, magnetic flux through one turn of wire

$$= BA = \frac{nA}{\varepsilon_0 c^2} I$$

In a length l of solenoid, there are nl turns of wire

$\Rightarrow$ magnetic flux in length l

$$\Phi = \frac{n^2 A l}{\varepsilon_0 c^2} I$$

$\Rightarrow$ $L = \frac{n^2 A l}{\varepsilon_0 c^2}$ for a solenoid of length l

- Going back to the general result $\Phi = L I$, if the current changes, Faraday's law

$$\Rightarrow \mathcal{E} = -\frac{d\Phi}{dt} = -L\frac{dI}{dt}$$

$$\boxed{\mathcal{E} = -L\frac{dI(t)}{dt}}$$

The minus sign is Lenz's law: the induced emf opposes the change in current e.g. if the current is increasing in a given direction, the induced emf is in the opposite direction.

For this reason, $\mathcal{E} = -L\frac{dI(t)}{dt}$ is sometimes called the "back emf".

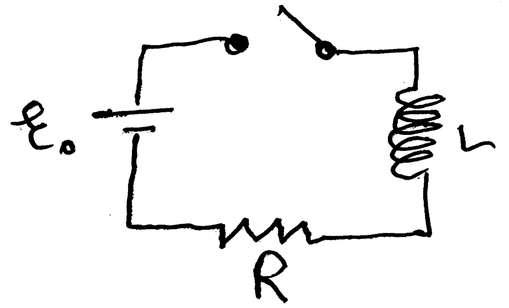
Example : If a current carrying wire is cut, the current $\to 0$ almost instantly

$\Rightarrow \frac{dI}{dt}$ enormous even if I is not

$\Rightarrow$ very large back emf

This is why you sometimes get a spark when you unplug an electrical device.

But when you plug the device in – inductance in the circuit opposes the increase in current $\Rightarrow$ steady buildup of current.



When we close the switch, total emf

$$= \underbrace{\mathcal{E}_0 - L\frac{dI(t)}{dt}}$$

"back emf", opposes current buildup

Equating this to the voltage drop $I(t)$ across the resistor, we get

$$\mathcal{E}_0 - L\frac{dI(t)}{dt} = I(t)R$$

$$\Rightarrow \quad \frac{dI(t)}{dt} = \frac{\mathcal{E}_0}{L} - \frac{R}{L}I(t)$$

This is a first order differential equation for the current as a function of time

- Solution : $I(t) = \frac{\mathcal{E}_0}{R} + \alpha e^{-\frac{R}{L}t}$,
α = constant.

[Check $\frac{dI(t)}{dt} = -\frac{R}{L}\alpha e^{-\frac{R}{L}t}$

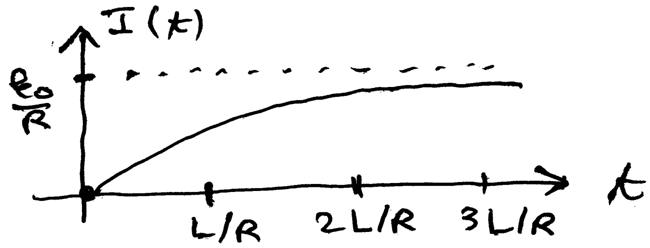
$$\frac{\mathcal{E}_0}{L} - \frac{R}{L}I(t) = \frac{\mathcal{E}_0}{L} - \left(\frac{\mathcal{E}_0}{L} + \frac{R}{L}\alpha e^{-\frac{R}{L}t}\right)$$
$$= -\frac{R}{L}\alpha e^{-\frac{R}{L}t}$$

- So LHS = RHS].

If the switch is closed at $t=0$,

$$I(0) = 0 \quad \Rightarrow \quad \frac{\mathcal{E}_0}{R} + \alpha = 0$$
$$\Rightarrow \quad \alpha = -\frac{\mathcal{E}_0}{R}$$

$$\Rightarrow I(t) = \frac{\mathcal{E}_0}{R} - \frac{\mathcal{E}_0}{R}e^{-\frac{R}{L}t}$$
$$= \frac{\mathcal{E}_0}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$



Current builds up gradually until

$$I = \frac{\mathcal{E}_0}{R}$$

ie $IR = \mathcal{E}_0$, have a steady flow of current through the resistor.

The rate of change of current is determined by the time constant $\tau = \frac{L}{R}$.

Energy in magnetic fields

It takes a certain amount of energy to start a current flowing in a circuit – we have to do work against the back emf. produced by inductance.

This energy is recoverable – it is stored in magnetic fields, and we get it back when we switch the current off. (but energy lost in a resistor is not recovered, it escapes as heat).

[Also – energy used in charging capacitors is recoverable, it is stored in the electric field, it is recovered when the capacitor discharges].

- The work done in moving a single charge q around the circuit against the back emf is $W = -\mathcal{E}q$

($\mathcal{E} q$ is the work done by the emf on the charge, $-\mathcal{E} q$ is the work you do).

Since I = charge passing a point per unit time

$$\frac{dW}{dt} = -\mathcal{E}\frac{dq}{dt}$$
$$= -\mathcal{E} I(t)$$
$$= L\frac{dI(t)}{dt}\cdot I(t)$$
$$= \frac{1}{2} L \frac{d}{dt} I(t)^2$$

If the current starts at 0 at time zero and builds I at time T

$$W = \int_0^T \frac{dW}{dt}\, dt = \frac{1}{2} L \int_0^T \left(\frac{d}{dt} I(t)^2\right) dt$$
$$= \frac{1}{2} L\, I(T)^2$$
$$= \frac{1}{2} L I^2.$$

This is stored as potential energy built up in the magnetic fields created by the current,

$$\boxed{U = \frac{1}{2} L I^2}$$

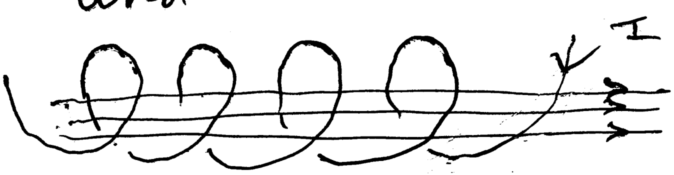
[Compare with $U = \frac{1}{2}\frac{Q^2}{C}$ for a capacitor. Why L in $U_{magnetic}$ and $\frac{1}{C}$ in $U_{electric}$?

Absence of magnetic energy $\Rightarrow L = 0$ as current must produce no magnetic flux

For a capacitor, absence of electric energy $\Rightarrow \Delta V = 0$ for a capacitor of charge Q.

But $\Delta V = \frac{1}{C} Q \Rightarrow \frac{1}{C} = 0$ for no potential difference $\Rightarrow C = \infty$].

Example: A solenoid with current I and n turns per unit length



$$B = \frac{nI}{\varepsilon_0 c^2} \qquad -(1)$$

Earlier we showed: $L = \frac{n^2 A l}{\varepsilon_0 c^2}$ for a solenoid of length l.

So magnetic potential energy

$$U = \frac{1}{2} L I^2$$