$$= \frac{1}{2} \frac{n^2 A l}{\varepsilon_0 c^2} I^2$$

But $I = \dfrac{\varepsilon_0 c^2 B}{n}$ from (1)

$$\Rightarrow U = \frac{1}{2} \frac{n^2 A l}{\varepsilon_0 c^2} \frac{(\varepsilon_0 c^2)^2 B^2}{n^2}$$

$$= \frac{1}{2} \varepsilon_0 c^2 B^2 A l.$$

- The volume occupied by the magnetic field is $A.l$ $\Rightarrow$ magnetic energy <u>density</u> is

$$u = \frac{U}{A.l}$$

$$= \frac{1}{2} \varepsilon_0 c^2 |\vec{B}|^2$$

- (Compare with energy density of an electric field

$$u = \frac{1}{2} \varepsilon_0 |\vec{E}|^2 \text{).}$$

$\uparrow$ <u>energy density</u>

- The result $u_{magnetic} = \frac{1}{2} \varepsilon_0 c^2 |\vec{B}|^2$ was derived for a special case, namely when the inductance is associated with a solenoid.

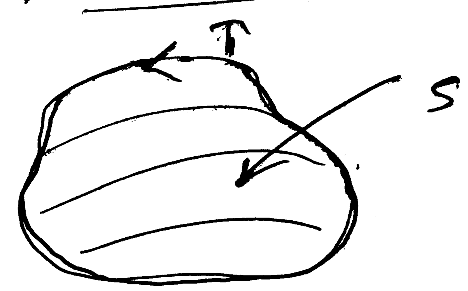
• However: this result applies for arbitrary magnetic fields

Outline of proof

Earlier, we defined the vector potential $\vec{A}(\vec{r})$ defined by $\vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r})$.

• this automatically solves the Maxwell equation $\vec{\nabla} \cdot \vec{B}(\vec{r}) = 0$ (as $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}(\vec{r})) = 0$ via an identity true for any vector field).

Then for a surface S with boundary



• T,

magnetic flux through S

$$\Phi = \int_S \vec{B} \cdot d\vec{S}$$

$$= \int_S (\vec{\nabla} \times \vec{A}) \cdot d\vec{S}$$

$$= \oint_T \vec{A} \cdot d\vec{l}$$ ← Stokes' theorem

↑ circulation of $\vec{A}$ around T.

But by definition of L (inductance due to current in T),

$$\Phi = LI$$

ie (flux through S) $= L \times$ (current in T)

ie $\oint_T \vec{A} \cdot d\vec{l} = L \times$ (current in T)

$$= LI.$$

But we showed the potential energy due to the magnetic field produced by I is

$$U = \tfrac{1}{2} L I^2$$

$$= \tfrac{1}{2} I . (LI)$$

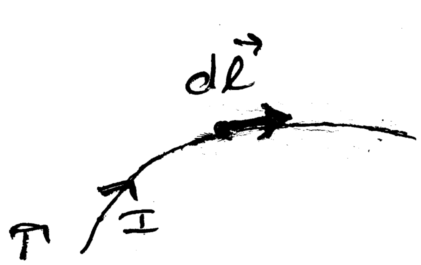
$$= \tfrac{1}{2} I \oint_T \vec{A} \cdot d\vec{l}$$

$$= \tfrac{1}{2} \oint_T \vec{A} \cdot I\, d\vec{l}$$

$$= \tfrac{1}{2} \oint_T \vec{A} \cdot \vec{I}\, dl$$

where $\vec{I} = I \dfrac{d\vec{l}}{dl}$

↑ unit vector tangent to loop T



• This was for a current I in a path Γ.

The generalization to a <u>current density</u> $\vec{j}(\vec{r})$ is

$$U_{magnetic} = \frac{1}{2}\int \vec{A}(\vec{r}) \cdot \underbrace{\vec{j}(\vec{r})\, d^3\vec{r}}_{\vec{I}\, dl,}$$

integral over all space

where $\vec{I}$ is current through surface $\perp$ to $\vec{j}$, and dl is distance in direction of $\vec{j}$.

But Ampere's law

$$\vec{\nabla} \times \vec{B}(\vec{r}) = \frac{\vec{j}(\vec{r})}{\varepsilon_0 c^2}$$

$$\Rightarrow \vec{j}(\vec{r}) = \varepsilon_0 c^2\, \vec{\nabla} \times \vec{B}(\vec{r})$$

So U magnetic

$$= \frac{\varepsilon_0 c^2}{2} \int \vec{A}(\vec{r}) \cdot (\vec{\nabla} \times \vec{B}(\vec{r}))\, d^3\vec{r}$$

Claim; using vector idenities,*
we can "integrate by parts" to
move "$\vec{\nabla}\times$" from acting on $\vec{B}$
to act on $\vec{A}$ (also need
$\vec{A}$ and $\vec{B} \to 0$ at infinity
surface terms at infinity vanish).

- Result:

U_{magnetic}

$$= \frac{\varepsilon_0 c^2}{2} \int \left(\vec{\nabla}\times\vec{A}(\vec{r})\right)\cdot\vec{B}(\vec{r})\, d^3\vec{r}$$

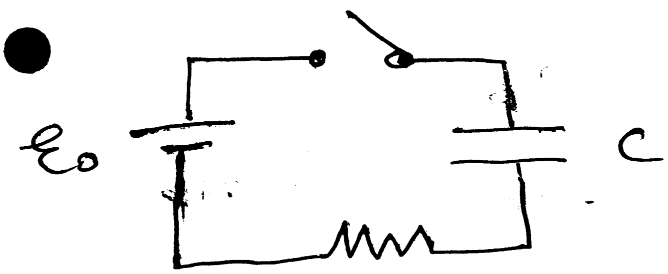
$$= \frac{\varepsilon_0 c^2}{2} \int \underbrace{\vec{B}(\vec{r})\cdot\vec{B}(\vec{r})}_{|\vec{B}(\vec{r})|^2}\, d^3\vec{r}$$

- So magnetic energy density
(magnetic energy per unit volume)

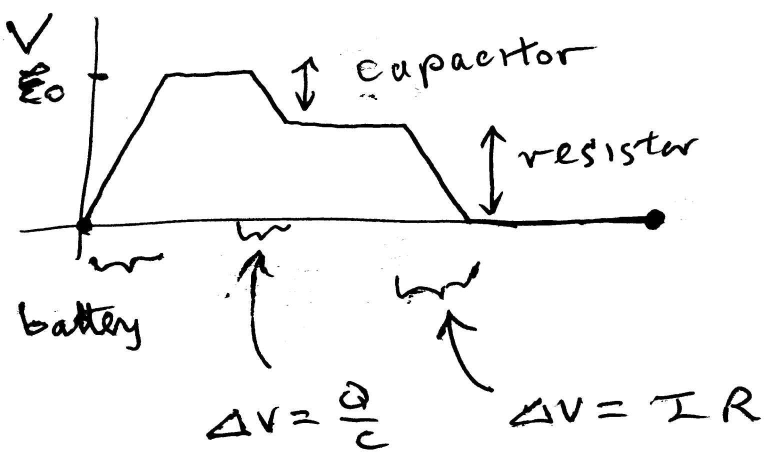
$$\boxed{u_{\text{magnetic}} = \frac{1}{2}\varepsilon_0 c^2 |\vec{B}(\vec{r})|^2}$$

* Vector identity
$\vec{A}\cdot(\vec{\nabla}\times\vec{B}) = -\vec{\nabla}\cdot(\vec{A}\times\vec{B}) + (\vec{\nabla}\times\vec{A})\cdot\vec{B}$

Examples of circuits with capacitors and inductors

- Earlier: look at an LR circuit (inductor and resistor).
- RC circuit (capacitor and resistor).
- 

When the switch is closed, the sum of the voltage changes around the circuit must be zero.



If we go around the circuit stating from negative terminal of battery

$$0 = \mathcal{E}_0 - \frac{Q}{C} - IR.$$

$$0 = \mathcal{E}_0 - \frac{1}{C} Q(t) - R \frac{dQ(t)}{dt}$$

$$\frac{dQ(t)}{dt} = \frac{\mathcal{E}_0}{R} - \frac{1}{RC} Q(t)$$

↑ first order d.e

Solution:

$$Q(t) = \mathcal{E}_0 C + \alpha e^{-\frac{t}{RC}}$$

↑ constant

[Check $\frac{dQ(t)}{dt} = -\frac{\alpha}{RC} e^{-t/RC}$

$$\frac{\mathcal{E}_0}{R} - \frac{1}{RC} Q(t)$$

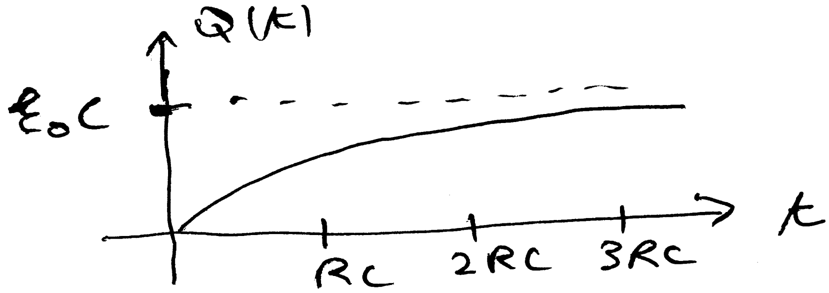
$$= \frac{\mathcal{E}_0}{R} - \frac{1}{RC} \left(\mathcal{E}_0 C + \alpha e^{-t/RC} \right)$$

$$= \cancel{\frac{\mathcal{E}_0}{R}} - \cancel{\frac{\mathcal{E}_0}{R}} - \frac{\alpha}{RC} e^{-t/RC} \quad].$$

If $Q(t) = 0$ at $t = 0$,

$$0 = \mathcal{E}_0 C + \alpha \Rightarrow \alpha = -\mathcal{E}_0 C$$

$$\Rightarrow \boxed{Q(t) = \mathcal{E}_0 C \left(1 - e^{-t/RC}\right)}$$



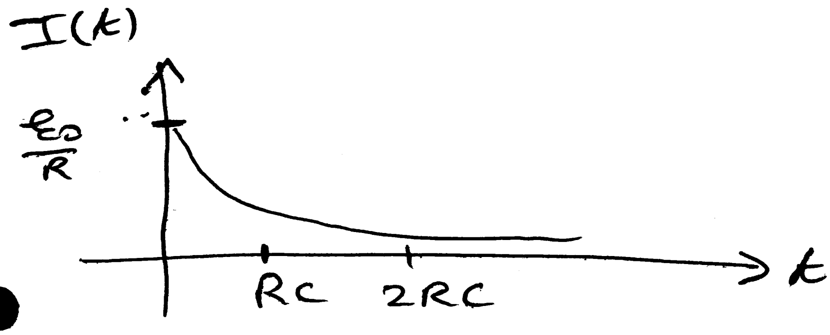
The capacitor charges up until

$$Q = \mathcal{E}_0 C \qquad (Q = C\Delta V)$$

- $$I(t) = \frac{dQ(t)}{dt}$$

$$= -(\mathcal{E}_0 C)\left(-\frac{1}{RC}\right) e^{-t/RC}$$

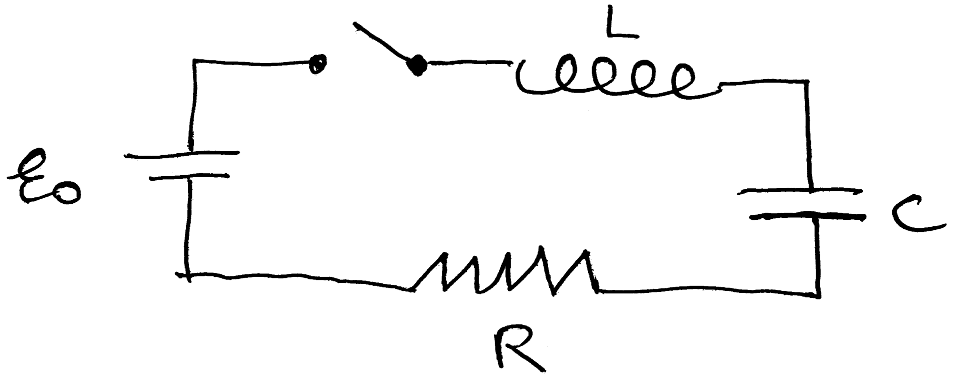
$$= \frac{\mathcal{E}_0}{R} e^{-t/RC}$$



Current dies off as capacitor charges

Time constant $\tau = RC$

Example: RLC circuit



- When the switch is closed,

$$0 = \mathcal{E}_0 - L\frac{dI}{dt} - \frac{Q}{C} - IR$$

emf of battery

back emf due to inductor

voltage drop across capacitor

voltage drop across resistor.

- Using $I(t) = \dfrac{dQ(t)}{dt}$

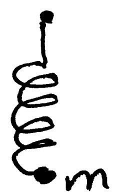
$$0 = \mathcal{E}_0 - L\frac{d^2Q(t)}{dt^2} - \frac{1}{C}Q(t) - R\frac{dQ(t)}{dt}$$

second order differential equation for $Q(t)$

$$\Rightarrow L\frac{d^2Q(t)}{dt^2} = \mathcal{E}_0 - R\frac{dQ(t)}{dt} - \frac{1}{C}Q(t).$$

where $Q(t)$ is the charge on the capacitor

- Mathematically, this takes the same form as that for a <u>damped simple harmonic oscillator</u> – a particle attached to a spring



If we displace the particle from equilibrium, by an amount x, $F = -kx$ ($-$ sign means restoring force)

$$\underbrace{m\frac{d^2x(t)}{dt^2}}_{ma} = \text{Force}$$

$$= F_0 - b\frac{dx(t)}{dt} - kx(t)$$

F_0: some constant force

$b\frac{dx(t)}{dt}$: damping term proportional to speed

$kx(t)$: spring restoring force

<u>So</u>: • $L\frac{d^2Q(t)}{dt^2} \leftrightarrow m\frac{d^2x(t)}{dt}$,

m is a measure of the resistance to a change of speed ($= \frac{dx(t)}{dt}$) for a particle when a force is applied,