TEST 2 - SOLUTIONS

1.

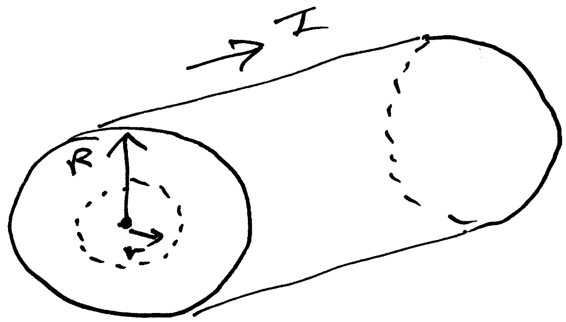
Ampere's law:

Circulation of magnetic field around Amperian loop $= \frac{1}{\varepsilon_0 c^2}$ (current enclosed)

ie $\quad 2\pi r \; B(r) = \frac{1}{\varepsilon_0 c^2} \; I$

$$\boxed{B(r) = \frac{I}{2\pi \varepsilon_0 c^2 \, r}}$$

2.



Current density

= current per unit cross-sectional area

$= \frac{I}{\pi R^2}$

Current through circle of radius r

= (current density) × area of circle

$$= \frac{I}{\pi R^2} \times \pi r^2$$

$$= \frac{I r^2}{R^2}$$

Ampere's law: (circulation of $\vec{B}$ field around Amperian loop) = $\frac{1}{\varepsilon_0 c^2}$ (current through loop)

$$\Rightarrow \quad 2\pi r \; B(r) = \frac{1}{\varepsilon_0 c^2} \frac{I r^2}{R^2}$$

$$\Rightarrow \quad \boxed{B(r) = \frac{I r}{2\pi \varepsilon_0 c^2 R^2}}$$

3. Magnetic flux through loop

$\Phi(t)$ = (magnitude of magnetic field) × (area of circle)

$$= B_0 t \times \pi R^2$$

$$|\mathcal{E}| = \left| \frac{d\Phi(t)}{dt} \right|$$

$$= \pi R^2 B_0$$

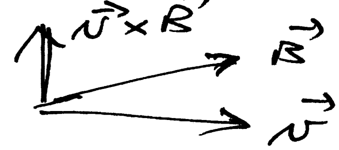
4. $\vec{F} = q\,\vec{v} \times \vec{B}$

Work done during displacement $d\vec{x}$

$= \vec{v}\,dt$ is

$$\vec{F} \cdot d\vec{x} = q\,(\vec{v} \times \vec{B}) \cdot \vec{v}\,dt.$$

<u>But</u>: $\vec{v} \times \vec{B}$ is perpendicular to $\vec{v}$ (and $\vec{B}$)

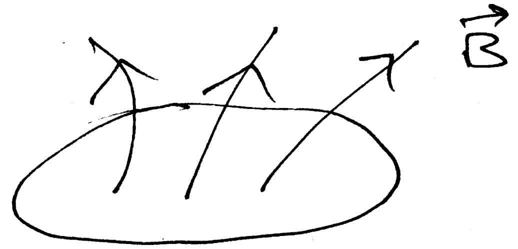


$$\Rightarrow (\vec{v} \times \vec{B}) \cdot \vec{v} = 0$$

$$\Rightarrow \vec{F} \cdot d\vec{x} = 0$$

ie the magnetic field does no work on the particle because the force is perpendicular to the displacement $\vec{v}\,dt$ that occurs in time dt

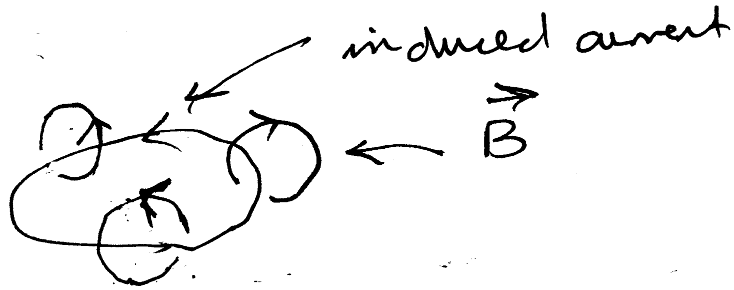
5.



Flux of $\vec{B}$ out of page decreasing

$\Rightarrow$ induced current must <u>increase</u> the flux of $\vec{B}$ out of the page

$\Rightarrow$ current is counter-clockwise



6.

Ampere's law

$\Rightarrow$ circulation around Γ

$= \frac{1}{\varepsilon_0 c^2} \times$ (current enclosed by Γ)

i.e. circulation around $\Gamma = \dfrac{I}{\varepsilon_0 c^2}$

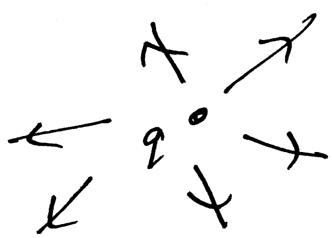
7. $\vec{\nabla} \cdot \vec{B} = 0$ (divergence of $\vec{B}$)

$\Rightarrow$ magnetic field lines are
not diverging or converging

$\Rightarrow$ no point "sources" or "sinks"
of magnetic fields

(contrast with $\vec{\nabla} \cdot \vec{E} = \rho / \varepsilon_0$

$\Rightarrow$ for a point electric charge



(DIVERGING FIELD))

8. $f(x,t) = A e^{i(k x - \omega t)}$

$$\frac{\partial^2 f(x,t)}{\partial x^2} = (ik)^2 A e^{i(kx - \omega t)}$$

$$\frac{1}{v^2} \frac{\partial^2 f(x,t)}{\partial t^2} = \frac{(-i\omega)^2}{v^2} A e^{i(kx - \omega t)}$$

Equating

$$-k^2 A e^{i(kx - \omega t)} = -\frac{\omega^2}{v^2} A e^{i(kx - \omega t)},$$

this is solved by $k = \frac{\omega}{v}$

9. $\vec{B}(\vec{r},t) = \vec{B}_0 \, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$

$= \vec{B}_0 \, e^{i(k_x x + k_y y + k_z z - \omega t)}$

$\vec{\nabla} \cdot \vec{B}(\vec{r},t)$

$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial \cancel{z}}\right) \vec{B}(\vec{r},t)$

$\frac{\partial}{\partial x}\left(\vec{B}_0 \, e^{i(\vec{k}\cdot\vec{r} - \omega t)}\right)$

$= i k_x \, \vec{B}_0 \, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$

$\frac{\partial}{\partial y}\left(\vec{B}_0 \, e^{i(\vec{k}\cdot\vec{r} - \omega t)}\right)$

$= i k_y \, \vec{B}_0 \, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$

$\frac{\partial}{\partial z}\left(\vec{B}_0 \, e^{i(\vec{k}\cdot\vec{r} - \omega t)}\right)$

$= i k_z \, \vec{B}_0 \, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$

So $\vec{\nabla} \cdot \vec{B}(\vec{r},t)$

$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \vec{B}_0 \, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$

$= (i k_x, i k_y, i k_z) \cdot \vec{B}_0 \, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$

$= i \vec{k} \cdot \vec{B}_0 \, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$

So $\vec{\nabla} \cdot \vec{B}(\vec{r}, t) = 0$

$$\Rightarrow \boxed{\vec{k} \cdot \vec{B_0} = 0}$$

10. $\oint_S \vec{J}(\vec{r}, t) \cdot d\vec{S}$

= net current through surface S

= rate of charge of current inside surface S.

If there is a net flow of current out through S, $\oint_S \vec{J}(\vec{r}, t) \cdot d\vec{S} > 0$.

The charge inside S, $\int_V \rho(\vec{r}, t)\, d^3\vec{r}$

is decreasing, and so $\frac{d}{dt} \int_V \rho(\vec{r}, t)\, d^3\vec{r} < 0$.

To match the signs

$$\oint_S \vec{J}(\vec{r}, t) \cdot d\vec{S} = -\frac{d}{dt} \int_V \rho(\vec{r}, t)\, d^3\vec{r}$$