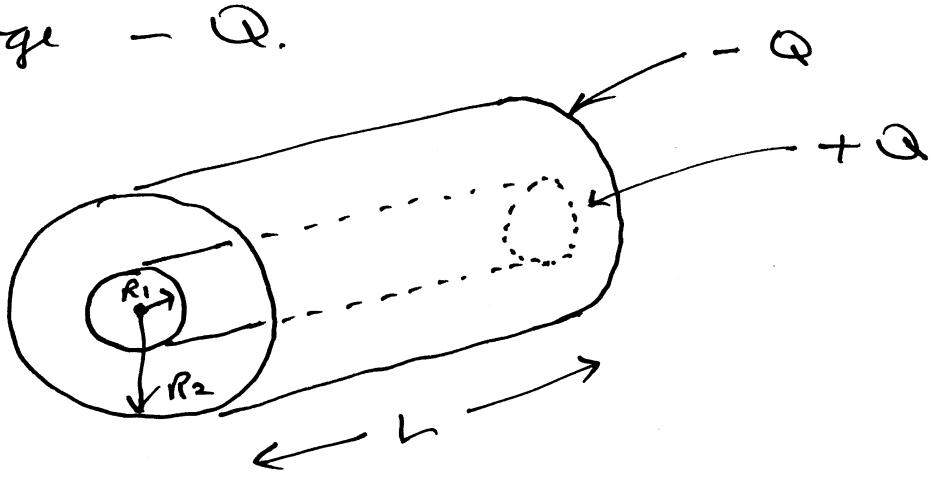
PROBLEM SET 10 — 2019

1. Consider a concentric pair of cylindrial ~~conductors~~ conducting surfaces of radius R_1 and R_2 and length L. Suppose the inner conductor has a charge $+Q$ and the outer conductor has charge $-Q$.



Due to the symmetry of the charge distribution, any electric field will be radial (inwards or out-wards). Due to the symmetry of the charge distribution, any electric field is radial (pointing inwards or outwards) and has a magnitude $E(r)$ that depends only on the radial distance r from the axis of the cylinders.

(a) By choosing an appropriate Gaussian surface and using Gauss's law, compute $E(r)$ and the direction of the electric field for

(i) $r < R_1$
(ii) $R_1 < r < R_2$
(iii) $r > R_2$

(b) Compute the potential difference ΔV between the inner and outer cylinders, and indicate which cylinder has the higher electric potential (Note $V(\vec{r}_2) - V(\vec{r}_1) = - \int_{T} \vec{E}(\vec{r}) \cdot d\vec{l}$ where T is a path from $\vec{r}_1$ to $\vec{r}_2$)

(c) What is the amount of work required to move a charge $+q$ from the outer cylinder to the inner one?

(d) Express $E(r)$ in terms of ΔV rather than $\frac{Q}{L}$.

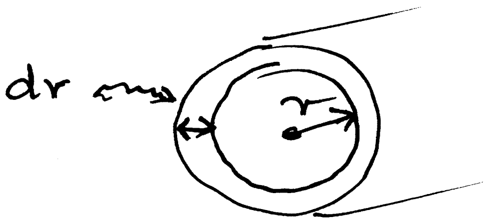
(e) Compute the capacitance C of the pair of cylinders using

$$\Delta V = \frac{1}{C} Q$$

(f) Determine the electric potential energy stored in the electric field

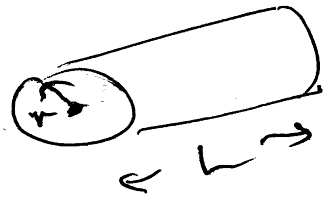
<u>Note</u>: energy density $u(r) = \frac{\varepsilon_0}{2} E(r)^2$.

First compute the energy in in thin cylindrical shell between r and $r+dr$, then integrate from $r = R_1$ to $r = R_2$



PROBLEM SET 10 - SOLUTIONS

1 (a). As Gaussian surface, choose a cylinder of radius r and length L centred on the axis of the pair of cylinders



Gauss law

$\Rightarrow$ (flux of $\vec{E}$ field through surface)

$$= \frac{(\text{charge enclosed})}{\varepsilon_0}.$$

ie. $2\pi r L\, E(r) = \dfrac{(\text{charge enclosed})}{\varepsilon_0}$

(i) For $r < R_1$, charge enclosed by Gaussian surface $= 0$

$\Rightarrow \quad 2\pi r L \; E(r) = 0$

$\Rightarrow \quad \boxed{E(r) = 0}$

(ii) For $R_1 < r < R_2$, charge enclosed by Gaussian surface $= +Q$

$$\Rightarrow \quad 2\pi r L\, E(r) = \frac{Q}{\varepsilon_0}$$

$$\Rightarrow \boxed{E(r) = \frac{Q}{L}\,\frac{1}{2\pi\varepsilon_0 r}}$$

The electric field points <u>outwards</u> in the radial direction between the cylinders

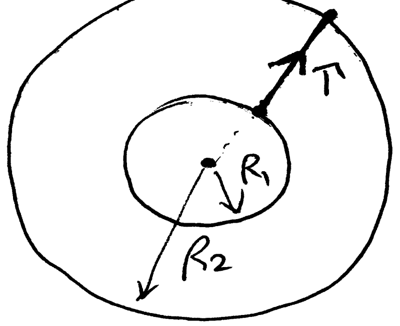
(iii) For $r > R_2$, charge ~~encl~~ enclosed by Gaussian surface $= +Q + (-Q)$
$= 0$

$$\Rightarrow \quad 2\pi r L\, E(r) = 0$$

$$\Rightarrow \boxed{E(r) = 0}$$

(b) $$V(\vec{r_2}) - V(\vec{r_1}) = -\int_{\Gamma} \vec{E}(\vec{r}) \cdot d\vec{l}$$

Choose a radial path from $r = R_1$ to $r = R_2$



$$E(r) = \frac{Q}{L}\,\frac{1}{2\pi\varepsilon_0 r}$$

$\vec{E}$ is outwards.

$$\Rightarrow \vec{E}(r)\cdot d\vec{l} = \frac{Q}{L}\,\frac{1}{2\pi\varepsilon_0 r}\,dr$$

$$\Rightarrow V(R_2) - V(R_1) = -\frac{Q}{L}\,\frac{1}{2\pi\varepsilon_0}\int_{R_1}^{R_2}\frac{dr}{r}$$

$$= -\frac{Q}{L}\,\frac{1}{2\pi\varepsilon_0}\left[\ln r\right]_{R_1}^{R_2}$$

$$= -\frac{Q}{L}\,\frac{1}{2\pi\varepsilon_0}\left(\ln R_2 - \ln R_1\right)$$

$$= -\frac{Q}{L}\,\frac{1}{2\pi\varepsilon_0}\ln\frac{R_2}{R_1}$$

$$< 0$$

So the outer cylinder has a lower electric potential than the inner one (which agrees with the fact that the $\vec{E}$ field points from the surface of higher electric potential to that with lower electric potential)

Alternatively, it takes work to move a charge $+q$ from the outer cylinder to the inner cylinder (as the force on the charge is outwards)

$\Rightarrow$ its potential energy increases

as we move from the outer cylinder

to the inner one

$\Rightarrow$ the electric potential (= potential energy per unit charge) is higher on the inner cylinder.

<u>So</u>:

$$\Delta V = \frac{Q}{L} \frac{1}{2\pi\varepsilon_0} \ln \frac{R_2}{R_1}$$

with the inner cylinder having the higher electric potential.

(c) ΔV = electric potential energy per unit charge.

So the change in electric potential energy in moving a charge $+q$ from the outer to the inner cylinder is

$$W = q\,\Delta V = q \frac{Q}{L} \frac{1}{2\pi\varepsilon_0} \ln \frac{R_2}{R_1}$$

(d) $$E(r) = \frac{Q}{L}\frac{1}{2\pi\varepsilon_0 r} \quad -① $$

$$\Delta V = \frac{Q}{L}\frac{1}{2\pi\varepsilon_0}\ln\frac{R_2}{R_1}$$

$$\Rightarrow \frac{Q}{L} = \frac{2\pi\varepsilon_0 \Delta V}{\left(\ln\frac{R_2}{R_1}\right)}$$

Substitute into ①

$$\boxed{E(r) = \frac{\Delta V}{\left(\ln\frac{R_2}{R_1}\right)}\frac{1}{r}}$$

(e). $$C = \frac{Q}{\Delta V}$$

$$= \frac{2\pi\varepsilon_0 L}{\left(\ln\frac{R_2}{R_1}\right)}.$$

(f). Energy density (energy per unit volume)

$$u(r) = \frac{\varepsilon_0}{2}E(r)^2$$

$$= \left(\frac{Q}{2\pi\varepsilon_0 L}\right)^2\frac{1}{r^2}$$

The volume between r and $r+dr$

$= 2\pi r . dr . L$

$\Rightarrow$ energy contained in the cylindrical shell between r and $r+dr$

$$= \frac{\varepsilon_0}{2} u(r)\, 2\pi r L\, dr$$

$$= \frac{\varepsilon_0}{2} \left(\frac{Q}{2\pi\varepsilon_0 L}\right)^2 \frac{1}{r^2}\; 2\pi r L\, dr$$

$$= \frac{Q^2}{4\pi\varepsilon_0 L} \frac{dr}{r}$$

$\Rightarrow$ energy between cylinder

$$= \frac{Q^2}{4\pi\varepsilon_0 L} \int_{R_1}^{R_2} \frac{dr}{r}$$

$$= \frac{Q^2}{4\pi\varepsilon_0 L} \left[\ln r\right]_{R_1}^{R_2}$$

$$= \frac{Q^2}{4\pi\varepsilon_0 L} (\ln R_2 - \ln R_1)$$

$$= \frac{Q^2}{4\pi\varepsilon_0 L} \ln \frac{R_2}{R_1}$$

[Note: this is the same as

$\frac{1}{2} C (\Delta V)^2$

Check:

$$\frac{1}{2} C (\Delta V)^2$$

$$= \frac{1}{2} \cdot \frac{2\pi \varepsilon_0 L}{\left(\ln \frac{R_2}{R_1}\right)} \cdot \frac{Q^2}{(2\pi \varepsilon_0 L)^2} \left(\ln \frac{R_2}{R_1}\right)^2$$

$$\doteq \frac{Q^2}{4\pi \varepsilon_0 L} \ln \frac{R_2}{R_1}$$